$O(2,1)$-like little group for spacelike four-momenta and localized light waves in continuous media

Y. S. Kim
Department of Physics, University of Maryland, College Park, Maryland 20742

Leehwa Yeh
Department of Physics, University of California at Berkeley, and Lawrence Berkeley Laboratory, Berkeley, California 94720

(Received 2 April 1992; accepted for publication 27 April 1992)

While a light wave travels in a nonvacuum medium with speed less than the speed of light in vacuum, its energy-momentum four-vector is spacelike. Thus the little group for this light wave is locally isomorphic to $O(2,1)$. It is shown that the transformation of this little group produces observable effects on the light wave, in contrast to the case of the $E(2)$-like little group for photons in vacuum. It is also shown that the little group can be defined for a localized light wave consisting of a superposition of waves with different frequencies traveling in the same direction.

I. INTRODUCTION

In his fundamental paper on representations of the Poincaré group,[1] Wigner observed that the four-vector of a free particle can be timelike, lightlike, or spacelike. He then defined the little group as the maximal subgroup of the Lorentz group, whose transformations leave the four-momentum vector invariant. The little groups for timelike, lightlike, and spacelike four-momenta are locally isomorphic to $O(3)$, $E(2)$, and $O(2,1)$, respectively.

The $O(3)$-like little group is associated with the spin of a massive particle at rest. The $E(2)$-like little group explains why massless particles have the helicity and gauge degrees of freedom. As for the $O(2,1)$-like little group, the traditional view is that particles with spacelike four-momentum travel faster than light. Thus, it has long been believed that this little group does not have any physical significance.

The purpose of the present paper is to point out that this view is not always correct, by working out an example. We consider here a light wave traveling in a continuous medium. As an example, let us consider the light wave going through a fishtank. The frequency is the same both inside and outside the tank. However, the wavelength is shorter in the tank, resulting in a speed less than the speed outside. In this case, the four-momentum is clearly spacelike. We shall examine this problem carefully in this paper.

Since the speed of the light wave in a continuous medium is smaller than c, there is a Lorentz frame in which the velocity is zero. This Lorentz frame is called the "zero-frequency" frame. In this frame, the Lorentz boost along the direction in the plane perpendicular to the momentum leaves the momentum invariant. In addition, the rotation around the momentum leaves it invariant. These transformations constitute the group $O(2,1)$, as Wigner observed in his paper.[1]

In this paper, we shall also consider the little-group transformation constructed from a boost preceded by rotation.[2,3] We start with a stationary fishtank, and assume that ϵ and μ of this medium are constant and isotropic, and therefore the index of refraction is also constant and isotropic. The rotation does not change the constitutive relations. However, the boost will make the fish tank move, and will change the constitutive relations.[4] We recall that, in vacuum, the translationlike little-group transformation on a propagating electromagnetic wave is a gauge transformation, and produces no observable effect on the electromagnetic wave.[5] We shall study how the moving fish tank will change this picture and will produce observable effects.

In Sec. II, it is noted that the light wave in a continuous medium travels with a spacelike four-momentum, but with the speed less than c. In Sec. III, we discuss the kinematics of the $O(2,1)$-like little group for the spacelike four-momentum. It is shown that, for a light wave in the continuous medium, there is a Lorentz frame in which the index of refraction is infinite, and the frequency vanishes. The little group in this frame is $O(2,1)$.

In Sec. IV, the $O(2,1)$-like transformation matrices is constructed as a boost preceded by a rotation, and is shown to be equivalent to Wigner's original form based on the zero-frequency frame. The transformation matrix leaves the spacelike four-momentum invariant. In Sec. V, we discuss the effect of the little-group transformations using the Lorentz transformation properties of electric field and magnetic induction, as well as the electric displacement and magnetic field. It is shown that the little-group transformations lead to observable effects. Section VI contains some illustrative examples.

 0022-2488/92/093190-07$03.00

In Sec. VII, it is noted that the little-group transformation matrix depends only on the index of refraction, but not on the frequency. This allows us to apply the same little group to light waves traveling in the same direction with different frequencies. It is thus possible to introduce the little group for localized light waves consisting of a superposition of waves with different frequencies. In Sec. VIII, we give a brief review of the efforts made to build the connection between Wigner's little groups and Maxwell's equations.

II. SPACELIKE MOMENTA

If a particle has a spacelike momentum, we tend to believe that its speed has to be greater than c, and therefore should not be observable. This is true when the energy-momentum relation is hyperbolic, but is not true when the relation is linear. If the energy-momentum relation is $E=(p^2+m^2)^{1/2}$, the particle velocity is

$$v=\frac{dE}{dp}=\frac{p}{E}. \tag{1}$$

This is smaller (greater) than one when m^2 is positive (negative). We use here the unit system in which $c=1$. On the other hand, if the energy-momentum relation is linear, as that of the photons traveling in vacuum, then the above reasoning is not valid. As for the photon traveling in a continuous medium along the z direction, we can write its four-potential as

$$A^\mu \exp\{i\omega(nz-t)\}, \tag{2}$$

where n is the index of refraction and is greater than or equal to 1. The energy and momentum are ω and $n\omega$, respectively. Indeed, the energy-momentum relation is still linear:

$$E=p/n, \tag{3}$$

and the speed is $1/n$. Since the momentum is greater than energy, the energy-momentum four-vector is spacelike.

Throughout this paper, we use the four-vector convention: $x^\mu=(x,y,z,t)$ and the metric $(+,+,+,-)$. Let us start with the four-vector A^μ of the form

$$A^\mu=(A_1,A_2,0,0), \tag{4}$$

in the laboratory frame in which the medium is at rest. Then this four-vector is invariant under boosts along the z direction. If we are on the train moving along the z direction with speed β, and if the train coordinate is specified by z' and t', then

$$z=(z'+\beta t')/\sqrt{1-\beta^2}, \quad t=(t'+\beta z')/\sqrt{1-\beta^2}. \tag{5}$$

To the observer on the train, the four-potential will appear as

$$A^\mu \exp\{i[\omega/(1-\beta^2)^{1/2}][(n-\beta)z'+(n\beta-1)t']\}. \tag{6}$$

If the train's speed reaches $\beta=1/n$, the above wave will become

$$A^\mu \exp\{i\omega(n^2-1)^{1/2}z'\}. \tag{7}$$

This happens in nature and causes the observable Cherenkov effect.

In this coordinate system, the traveling light wave will appear as a static sinusoidal wave like the static curve we see on the oscilloscope screen. We are interested in the symmetry transformations of this wave and their effects on the wave observed from the laboratory frame.

III. KINEMATICS OF THE LITTLE GROUP

Wigner's little group is the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. Let us start with a massive particle whose four-momentum is

$$P^\mu=m(0,0,\beta/(1-\beta^2)^{1/2},1/(1-\beta^2)^{1/2}). \tag{8}$$

Then it is possible to find the Lorentz frame in which the four-momentum is

$$P'^\mu=(0,0,0,m). \tag{9}$$

The transformation matrix which boosts the above four-momentum to that of Eq. (8) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/(1-\beta^2)^{1/2} & \beta/(1-\beta^2)^{1/2} \\ 0 & 0 & \beta/(1-\beta^2)^{1/2} & 1/(1-\beta^2)^{1/2} \end{pmatrix}. \tag{10}$$

It is quite clear that the maximal subgroup of the Lorentz group that leaves the four-momentum of Eq. (9) invariant is the three-dimensional rotation group. While leaving the momentum invariant, the little group changes the direction of the spin. The little group for the four-momentum of Eq. (8) is clearly a Lorentz-boosted rotation group.[5]

It is not difficult to visualize the symmetry of the $O(3)$-like little group, especially in the Lorentz frame in which the particle is at rest. Indeed, the $O(3)$-like little group for the massive particle has been exhaustively discussed in the literature in connection with the symmetry problems of elementary particles, Wigner rotation, and with the Thomas precession. It has also been established that the $E(2)$-like little group for massless particles as an infinite-momentum/zero-mass limit of the $O(3)$-like little group.

In this paper, we are interested in light waves traveling in a continuous nonvacuum medium, whose energy-momentum relation is different from that of the massive particle. As is explained in Sec. II, we start with the four-momentum

$$P^{\mu}=\omega(0,0,n,1). \tag{11}$$

Then there is a Lorentz frame in which the four-momentum becomes

$$P'^{\mu}=\omega(0,0,\sqrt{n^2-1},0). \tag{12}$$

This Lorentz frame is called the zero-frequency frame. It is possible to obtain the spacelike four-momentum of Eq. (11) from P'^{μ} of the zero-frequency frame by applying the boost matrix

$$L(n)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n/(n^2-1)^{1/2} & 1/(n^2-1)^{1/2} \\ 0 & 0 & 1/(n^2-1)^{1/2} & n/(n^2-1)^{1/2} \end{pmatrix}. \tag{13}$$

In the zero-frequency frame, the four-momentum P'^{μ} remains invariant under the rotation around the z axis. It is invariant also under the boosts along the x or y direction. Successive applications of these boosts result in a boost along the direction in the xy plane and rotation around the z axis. Indeed, Wigner's little group is the $(2+1)$-dimensional rotation group generated by the rotation generator

$$J_3=\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

and the boost generators

$$K_1=\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2=\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

They satisfy the commutation relations

$$[K_1,K_2]=-iJ_3, \quad [K_1,J_3]=-iK_2, \quad [K_2,J_1]=iJ_2, \tag{16}$$

for the group $O(2,1)$.

IV. LITTLE-GROUP TRANSFORMATIONS

The Lorentz boost along the x direction is generated by K_1 of Eq. (15), and the transformation matrix takes the form

$$F(\lambda)=\begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}. \tag{17}$$

This is one of the transformation matrices of Wigner's little group in the zero-frequency frame. This matrix leaves the four-momentum P'^{μ} of Eq. (12) invariant, but it transforms the four-vector A^{μ} of Eq. (4) into

$$A'^{\mu}=(A_1\cosh\lambda, A_2, 0, A_1\sinh\lambda). \tag{18}$$

From this, we can construct the little-group transformation matrix in the original frame, where the four-momentum is P^{μ}. The transformation matrix is

$$D(n,\lambda)=L(n)F(\lambda)[L(n)]^{-1}. \tag{19}$$

The matrix $[L(n)]^{-1}$ brings the four-vector to the zero-frequency frame, while transforming P^{μ} of Eq. (11) into P'^{μ} of Eq. (12), but leaving A^{μ} invariant. The boost $F(\lambda)$ leaves the four-momentum P'^{μ} invariant, but changes A^{μ} to A'^{μ}. Finally, $L(n)$ brings P'^{μ} back to P^{μ} and changes A'^{μ} to a new four-vector A''^{μ}. In the case of vacuum electrodynamics, A''^{μ} is a gauge transformation of A^{μ}. In the case of continuous media, the problem will be more complicated. This is what we propose to study in this paper.

After matrix multiplication, we can write $D(n,\lambda)$ as

$$D(n,\lambda)=\begin{pmatrix} \cosh\lambda & 0 & -b(\sinh\lambda) & bn(\sinh\lambda) \\ 0 & 1 & 0 & 0 \\ b(\sinh\lambda) & 0 & b^2(n^2-\cosh\lambda) & b^2n(\cosh\lambda-1) \\ bn(\sinh\lambda) & 0 & b^2n(1-\cosh\lambda) & b^2(n^2\cosh\lambda-1) \end{pmatrix}, \tag{20}$$

where $b = 1/\sqrt{n^2-1}$.

Let us now consider the matrix $D(n,\lambda)$ of Eq. (20) in the limit where the index of refraction becomes 1. In this case, the matrix becomes singular. However, we can take a controlled limit by making λ small, while holding

$$u=(\sinh\lambda)/\sqrt{n^2-1}, \tag{21}$$

finite. Then the D matrix becomes[3]

$$D(u)=\begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1-u^2/2 & u^2/2 \\ u & 0 & -u^2/2 & 1+u^2/2 \end{pmatrix}. \tag{22}$$

This is a transformation matrix of the $E(2)$-like little group for massless particles in vacuum.

The zero-frequency frame is very convenient for us to see that the little group is locally isomorphic to $O(2,1)$. On the other hand, it is possible to construct little-group transformation matrices without resorting to the zero-frequency frame. We are interested in the kinematics consisting of one Lorentz boost. Since rotations change only the direction of the momentum, it is possible to represent a transformation of the little group by a rotation followed by an energy-preserving boost that will bring back the four-momentum to its original form, as Kupersztych did for photons.[2] It was shown that this transformation is also possible for massive and imaginary-mass particles.[3] We shall see that this is also possible for light waves in a continuous medium.

Let us rotate the four-momentum of Eq. (11) around the y axis counterclockwise using the matrix

$$R(\theta)=\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

Then the four-momentum becomes

$$P'=\omega(n\sin\theta,0,n\cos\theta,1). \tag{24}$$

Next, it is possible to bring this to the original form of Eq. (11) by applying the boost matrix:[3]

$$S(n,\theta)=\begin{pmatrix} 1+2\sinh^2(\eta/2)\cos^2(\theta/2) & 0 & -\sinh^2(\eta/2)\sin\theta & -\sinh\eta\cos(\theta/2) \\ 0 & 1 & 0 & 0 \\ -\sinh^2(\eta/2)\sin\theta & 0 & 1+2\sinh^2(\eta/2)\sin^2(\theta/2) & \sinh\eta\sin(\theta/2) \\ -\sinh\eta\cos(\theta/2) & 0 & \sinh\eta\sin(\theta/2) & \cosh\eta \end{pmatrix}, \tag{25}$$

with $\tanh(\eta/2)=n\sin(\theta/2)$, according to which the absolute value of $\sin(\theta/2)$ cannot be greater than $1/n$.

Indeed, the rotation $R(\theta)$ followed by the boost $S(n,\theta)$ leaves the four-momentum P of Eq. (11) invariant. As a consequence, we should be able to write the D matrix as $D(n,\theta)=S(n,\theta)R(\theta)$. Indeed, the D matrix can also be written as

$$D(n,\theta)=\begin{pmatrix} -1+2/T & 0 & -u/T & nu/T \\ 0 & 1 & 0 & 0 \\ u/T & 0 & 1-u^2/2T & nu^2/2T \\ nu/T & 0 & -nu^2/2T & 1+nu^2/2T \end{pmatrix}, \tag{26}$$

where $u=-2(\tan(\theta/2))$, $T=1+(1-n^2)\tan^2(\theta/2)$. The above form becomes the D matrix of Eq. (20) if the angle θ is related to the boost parameter λ in the zero-frequency frame through the relation

$$\tanh\lambda=\frac{-2\sqrt{n^2-1}\tan(\theta/2)}{1+(n^2-1)\tan^2(\theta/2)}. \tag{27}$$

By definition, the D matrix of Eq. (26) does not change the four-momentum. In the case of vacuum electrodynamics, the matrix of Eq. (22) performs a gauge transformation on the four-potential that leads to no observable effects on electric and magnetic fields.[2,3] In the case of continuous medium, we have to deal with a moving continuous medium, and this will lead to observable effects. For this reason, in Sec. V, we shall work directly with the electric and magnetic fields, which are observable quantities.

V. OBSERVABLE EFFECTS OF THE LITTLE-GROUP TRANSFORMATIONS

It was noted in Sec. IV that the D matrix consists of the rotation $R(\theta)$ of Eq. (23) followed by the boost $S(n,\theta)$ of Eq. (25). The rotation does not change the index of refraction, but the boost may make it anisotropic.[4] Indeed, the boost is the essential part of the present problem. In order to cope with this situation, we can choose a coordinate system where the $S(n,\theta)$ matrix becomes a boost along the z direction.

The Lorentz boost $S(n,\theta)$ is along the $(\theta/2-90°)$ direction in the zx plane, while the four-momentum P is along the z direction. In order to simplify the problem, we can rotate the entire system counterclockwise by $(90° -\theta/2)$. Under this rotation, the rotation matrix $R(\theta)$ remains invariant, while the four-momentum vector becomes

$$P=\omega(n\cos(\theta/2),0,n\sin(\theta/2),1). \tag{28}$$

The rotation matrix $R(\theta)$ transforms this four-vector to

$$P'=\omega(n\cos(\theta/2),0,-n\sin(\theta/2),1). \tag{29}$$

The boost matrix $S(n,\theta)$ now takes the form

$$S(n)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\eta & \sinh\eta \\ 0 & 0 & \sinh\eta & \cosh\eta \end{pmatrix}. \tag{30}$$

This matrix performs a boost along the z direction, and brings the four-momentum P' of Eq. (29) into the original form of Eq. (28).

Before proceeding with waves in a continuous medium, let us check the validity of the above kinematics for vacuum electrodynamics with $n=1$. In this case, the rotation $R(\theta)$ of the wave propagating with the four-momentum P of Eq. (28), followed by the boost $S(n)$, is a gauge transformation.[3] Thus, the Lorentz boost $S(n)$ of the same wave propagating with the four-momentum P' of Eq. (29) should result in the original wave. Let us assume that the electric field is in the zx plane, and the magnetic field is in the y direction with unit magnitude. In the three-vector notation, the magnetic field is

$$\mathbf{B}=\mathbf{B}'=(0,1,0), \tag{31}$$

for both the four-momenta P and P'. The electric fields are

$$\begin{aligned} \mathbf{E}&=(\sin(\theta/2),0,-\cos(\theta/2)), \\ \mathbf{E}'&=(-\sin(\theta/2),0,-\cos(\theta/2)), \end{aligned} \tag{32}$$

for P and P', respectively. This is the result of the rotation $R(\theta)$. The question then is whether we can recover $\mathbf{E}$ and $\mathbf{B}$ by making a tensor transformation on $\mathbf{E}'$ and $\mathbf{B}'$ corresponding to the boost matrix of Eq. (30). The answer to this question is YES. This is a straightforward calculation, but is a necessary step before starting a more complicated computation for a continuous medium.

When the nonvacuum medium is stationary, the $\mathbf{E}$ and $\mathbf{D}$ are related by $\mathbf{D}=\epsilon\mathbf{E}$, and $\mathbf{H}$ and $\mathbf{B}$ by $\mathbf{B}=\mu\mathbf{H}$. The rotation of the system by $R(\theta)$ does not change these constitutive relations. Under the rotation, the y components of the fields are not affected, and the x and z components can be calculated from

$$\begin{pmatrix} E'_z \\ E'_x \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} E_z \\ E_x \end{pmatrix}, \tag{33}$$

and similar formulas for $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$.

Next, in order to complete the little-group transformation, we should boost $\mathbf{E}'$, $\mathbf{D}'$, $\mathbf{H}'$, and $\mathbf{B}'$ to $\mathbf{E}''$, $\mathbf{D}''$, $\mathbf{H}''$, and $\mathbf{B}''$, corresponding to the boost matrix of Eq. (30). It is important to realize that the medium is moving. Yet, the longitudinal components remain invariant, and thus

$$D''_z=\epsilon E''_z, \quad B''_z=\mu H''_z. \tag{34}$$

On the other hand, for the x and y components, the transformation law leads to

$$\begin{aligned} \mathbf{E}''_\perp&=(\cosh\eta)\mathbf{E}'_\perp-(\sinh\eta)\mathbf{k}\times\mathbf{B}'_\perp, \\ \mathbf{B}''_\perp&=(\cosh\eta)\mathbf{B}'_\perp+(\sinh\eta)\mathbf{k}\times\mathbf{E}'_\perp, \\ \mathbf{D}''_\perp&=(\cosh\eta)\mathbf{D}'_\perp-(\sinh\eta)\mathbf{k}\times\mathbf{H}'_\perp, \\ \mathbf{H}''_\perp&=(\cosh\eta)\mathbf{H}'_\perp+(\sinh\eta)\mathbf{k}\times\mathbf{D}'_\perp, \end{aligned} \tag{35}$$

where $\mathbf{k}$ is the unit vector along the z direction. Since $\mathbf{D}'=\epsilon\mathbf{E}'$, and $\mathbf{B}'=\mu\mathbf{H}'$,

$$\begin{aligned} \mathbf{D}''_\perp+(\tanh\eta)\mathbf{k}\times\mathbf{H}''_\perp&=\epsilon[\mathbf{E}''_\perp+(\tanh\eta)\mathbf{k}\times\mathbf{B}''_\perp], \\ \mathbf{B}''_\perp-(\tanh\eta)\mathbf{k}\times\mathbf{E}''_\perp&=\mu[\mathbf{H}''_\perp-(\tanh\eta)\mathbf{k}\times\mathbf{D}''_\perp]. \end{aligned} \tag{36}$$

These equations lead to the following constitutive relations:[4]

$$\begin{aligned} \mathbf{D}''_\perp&=\frac{\epsilon(1-\tanh^2\eta)}{1-n^2\tanh^2\eta}\mathbf{E}''_\perp+\frac{(n^2-1)\tanh\eta}{1-n^2\tanh^2\eta}\mathbf{k}\times\mathbf{H}''_\perp, \\ \mathbf{B}''_\perp&=\frac{\mu(1-\tanh^2\eta)}{1-n^2\tanh^2\eta}\mathbf{H}''_\perp-\frac{(n^2-1)\tanh\eta}{1-n^2\tanh^2\eta}\mathbf{k}\times\mathbf{E}''_\perp, \end{aligned} \tag{37}$$

which become $\mathbf{D}''_\perp=\epsilon\mathbf{E}''_\perp$ and $\mathbf{B}''_\perp=\mu\mathbf{H}''_\perp$, respectively, when $n=1$ and/or $\eta=0$.

VI. ILLUSTRATIVE EXAMPLES

In order to gain a clear picture of the little-group transformation, let us work out the following example. Initially, the wave propagates along the $(\cos(\theta/2), 0,\sin(\theta/2))$ direction, as is indicated in Eq. (28). Let us assume that the magnetic field **H** is in the y direction. Then the direction of the electric field **E** is $(\sin(\theta/2),0,-\cos(\theta/2))$ or

$$E_x=E\sin(\theta/2),\quad E_y=0,\quad E_z=-E\cos(\theta/2). \quad (38)$$

Under the rotation $R(\theta)$, the magnetic field and induction remain unchanged:

$$\mathbf{H}'=\mathbf{H},\quad \mathbf{B}'=\mathbf{B}. \quad (39)$$

Both $\mathbf{H}'$ and $\mathbf{B}'$ are in the y direction. The y and z components of the electric field also remain unchanged, but the x component changes its sign. Thus

$$E'_x=-E\sin(\theta/2),\quad E'_y=0,\quad E'_z=-E\cos(\theta/2). \quad (40)$$

Since the rotation does not affect the constitutive relations,

$$\mathbf{D}'=\epsilon\mathbf{E}',\quad \mathbf{B}'=\mu\mathbf{H}'. \quad (41)$$

It is now possible to complete the little-group transformation by boosting $\mathbf{E}'$, $\mathbf{B}'$, $\mathbf{D}'$, and $\mathbf{H}'$ to $\mathbf{E}''$, $\mathbf{B}''$, $\mathbf{D}''$, and $\mathbf{H}''$, respectively, using the transformation equations given in Eq. (34) and Eq. (35). The z components of the fields remain invariant, and

$$H''_z=B''_z=H''_x=B''_x=0 \quad (42)$$

and

$$E''_z=E_z,\quad D''_z=D_z,\quad D''_z=\epsilon E_z. \quad (43)$$

The transverse component of the electric field is determined by the first equation of Eq. (35). $\mathbf{E}'_\perp$ is known, and $\mathbf{B}'_\perp$ is the same as **B**, which can be determined from **E**. Both $E_\perp$ and $E''_\perp$ are in the x direction, with $E''_y = E_y = 0$ and

$$E''_x=\{n^2+(n^2-1)\cosh\eta\}E_x. \quad (44)$$

We can develop a similar reasoning for B''_y, and obtain

$$B''_y=\{n^{-2}+(1-n^{-2})\cosh\eta\}B_y. \quad (45)$$

As for D''_x and H''_y,

$$D''_x=D_x,\quad H''_y=H_y. \quad (46)$$

The little-group transformation changes E_x and B_y to E''_x and B''_y, respectively, producing an observable effect. However, it leaves D_x and H_x invariant, leaving **D** and **H** invariant. While the little-group transformation is a gauge transformation in vacuum electrodynamics, leaving also **E** and **B**, the transformation leaves invariant only D_x and H_x in the present case.

On the other hand, we arrive at a different conclusion if we can carry out a similar calculation for the case when the electric field is in the y direction, with $E_x=E_z=0$ and $D_x=D_z=0$. As for the magnetic field, $H_y=0$, and

$$H_x=-H\sin(\theta/2),\quad H_z=H\cos(\theta/2). \quad (47)$$

As in the previous case, the z component of the fields are not affected by the little-group transformation. According to the calculation, **E** and **B** remain invariant under the transformation. However,

$$D''_y=\{n^{-2}+(1-n^{-2})\cosh\eta\}D_y,$$
$$H''_x=\{n^2+(n^2-1)\cosh\eta\}H_x. \quad (48)$$

In this case, the remnant of the gauge invariance in vacuum electrodynamics appears as the invariance of **E** and **B**. However, the transformation still produces an observable effect through the Poynting vector.

We have carried out illustrative calculations for transverse waves in a initial stationary medium. The first calculation was for the case where the magnetic field is in the y direction, and the second for the electric field in the y direction. The most general case is a linear superposition of these two waves, since the electromagnetic wave in a stationary medium is always transverse.

VII. LOCALIZED LIGHT WAVES

Wigner's little group is based on a given value of four-momentum that corresponds in our case to a sharply defined frequency. However, the transformation matrix $D(n,\lambda)$ of Eq. (20) or Eq. (26) does not depend on the frequency ω. This means that the waves traveling along the z axis share the same little group as long as the index of refraction remains constant.

This allows us to define the little group for superposition of waves with different frequencies. For each component of the electromagnetic field, we can consider a localized wave of the form

$$\psi(z)=\int g(\omega)\exp(i\omega(nz-t))d\omega. \quad (49)$$

The little group transformation discussed in Sec. V is applicable without modification to this form of wave packet. The only change in Eq. (49) under the little group transformation is the change of its amplitude according to those formulas derived in Sec. VI.

VIII. CONCLUDING REMARKS

Wigner's fundamental paper on the Poincaré group was published more than 50 years ago.[1] This paper contains the relativistic interpretation of the spin of massive particles. However, it took many years for younger physicists to find the connection between Wigner's representation theory and Maxwell's equations.[2,5–8] The present paper is part of this continuing effort.

For massive particles, the physical implication of the little group is well known. For massless particles in vacuum, the little group is isomorphic to the two-dimensional Euclidean group. The translationlike degrees of freedom are gauge degrees of freedom. It has also been demonstrated that the $E(2)$-like little group for massless particles is an infinite-momentum/zero-mass limit of the $O(3)$-like little group.[3,9]

There is also the $O(2,1)$-like little group for spacelike momenta in Wigner's representation theory. It has long been thought that this little group is applicable to particles with speed greater than c with an imaginary mass. There are no physically meaningful representations of this little group applicable to such particles.[10]

It is interesting to note that the localized light wave in a continuous medium can be treated as a particle moving slower than light, but with a spacelike momentum. The $O(2,1)$-like little group is applicable to this case, and we discussed consequences of the little-group transformation.

[1] E. P. Wigner, Ann. Math. **40**, 149 (1939); V. Bargmann and E. P. Wigner, Proc. Nat'l. Acad. Sci. (USA) **34**, 211 (1948); E. P. Wigner, in *Theoretical Physics*, edited by A. Salam (International Atomic Energy Agency, Vienna, 1963).
[2] J. Kuperzstych, Nuovo Cimento **31B**, 1 (1976); Phys. Rev. D **17**, 629 (1978).
[3] D. Han, Y. S. Kim, and D. Son, J. Math. Phys. **27**, 2228 (1986).
[4] F. Hartemann and Z. Toffano, Phys. Rev. A **41**, 5066 (1990).
[5] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).
[6] S. Weinberg, Phys. Rev. B **134**, 882 (1964); **135**, 1049 (1964).
[7] A. Janner and T. Janssen, Physica **60**, 292 (1972).
[8] J. J. van der Bij, H. van Dam, and Y. J. Ng, Physica A **116**, 307 (1982).
[9] Y. S. Kim and E. P. Wigner, J. Math. Phys. **31**, 55 (1990).
[10] H. van Dam, Y. J. Ng, and L. C. Biedenharn, Phys. Lett. B **158**, 227 (1985).