Physics Letters A 162 (1992) 294–298
North-Holland

PHYSICS LETTERS A

Generalization of the integration-within-ordered-product technique to n-mode operator ordering ☆

Leehwa Yeh
Department of Physics, University of California and Physics Division, Lawrence Berkeley Laboratory, 1 Cyclotron Road, Berkeley, CA 94720, USA

Received 4 November 1991; accepted for publication 13 December 1991
Communicated by J.P. Vigier

Integration within an ordered product (IWOP) is a technique for calculating normally ordered forms of unitary operators which induce symplectic transformations. This technique is modified and generalized to general n-mode cases.

1. Introduction

In the past few years, a new approach for calculating normally ordered forms of some unitary operators has been developed by Fan and his collaborators, namely, the method of integration within an ordered product (IWOP) [1]. It has been shown that this approach is able to deal with those one-mode and two-mode operators that induce homogeneous symplectic transformations, i.e., the quantum mechanical analogue of classical homogeneous linear canonical transformations, or equivalently, those operators that exponentiate anti-Hermitian homogeneous quadratic forms of boson or fermion creation and annihilation operators.

The IWOP algorithm for solving the ordering problem is as follows: suppose U is an unitary operator which satisfies the above requirements, then it can be represented by the integration over projection operators in the coherent-state basis, e.g., in the one-mode case:

$$U \propto \int_{-\infty}^{\infty} \mathrm{d}^2(\alpha)\, |\alpha'\rangle\langle\alpha| ,$$

where α' represents the transformed α under the canonical transformation induced by U. This representation of U is "physically appealing" since it shows explicitly how the classical canonical transformations map to quantum unitary operators. Using properties of coherent states, we can express the integrand as a function of creation and annihilation operators within the normal ordering symbol. Since the orders of operators within a normal ordering symbol can be ignored, we can perform the integration by treating those creation and annihilation operators as c-numbers. The result of this integration is the normally ordered form of U.

Although the IWOP approach to the ordering problem is powerful and has been applied to many practical problems, there are some limitations. Firstly, it has always been taken as an ansatz whose validity had to be checked case by case. Because a general proof has never been given – the main difficulty being due to the overcompleteness of coherent states – this approach has only been applied to one-mode and two-mode prob-

☆ This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the US Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139.

lems, apart from a few special multimode cases. Secondly, after we perform IWOP, there remains a proportionality constant to be calculated by some auxiliary method; the phase of this constant has never been determined.

In order to remove these limitations, I have achieved a general proof of the applicability of IWOP to the ordering problem. Since this proof does not depend on the number of degrees of freedom, the validity of IWOP in general n-mode cases is established. As a by-product, the phase of the proportionality constant is also determined.

The proof given here only deals with boson cases, the proof for fermion cases is similar except that we have to use Grassmann numbers and fermion coherent states instead of ordinary complex numbers and canonical coherent states [2].

2. Representation of n-mode unitary operators

Let $\boldsymbol{q}=(q_1, q_2, ..., q_n)$ and $\boldsymbol{p}=(p_1, p_2, ..., p_n)$ be the canonical coordinate and momentum variables, $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{p}}$ being the operators which correspond to $\boldsymbol{q}$ and $\boldsymbol{p}$, $\boldsymbol{a}=(1/\sqrt{2})(\hat{\boldsymbol{q}}+\mathrm{i}\hat{\boldsymbol{p}})$ and $\hat{\boldsymbol{a}}^\dagger=(1/\sqrt{2})(\hat{\boldsymbol{q}}-\mathrm{i}\hat{\boldsymbol{p}})$ being the creation and annihilation operators, and $\boldsymbol{\alpha}=(1/\sqrt{2})(\boldsymbol{q}+\mathrm{i}\boldsymbol{p})$ and $\boldsymbol{\alpha}^*(1/\sqrt{2})(\boldsymbol{q}-\mathrm{i}\boldsymbol{p})$ being the classical correspondents of $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$. Throughout this paper all vectors are taken as *column* vectors. The Einstein summation convention is followed with dummy indices running from 1 to n. The asterisk denotes complex conjugate and the dagger denotes Hermitian conjugate.

Define the unitary operator U which induces a symplectic transformation as: $U\equiv\exp(\mathrm{i}H)$, where H is an n-mode homogeneous quadratic form made of $(\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}})$ or $(\boldsymbol{a}, \boldsymbol{a}^\dagger)$ and $H^\dagger=H$, i.e.,

$$H=\tfrac{1}{2}[\mu_{ij}p_ip_j+\nu_{ij}q_iq_j+\gamma_{ij}(p_iq_j+q_jp_i)]=\tfrac{1}{2}[K_{ij}a_i^\dagger a_j^\dagger+K_{ij}^*a_ia_j+\Lambda_{ij}(a_i^\dagger a_j+a_ja_i^\dagger)]\,, \tag{2.1}$$

where μ_{ij}, ν_{ij} and γ_{ij} are real, K_{ij} and Λ_{ij} are complex, $\mu_{ij}=\mu_{ji}$, $\nu_{ij}=\nu_{ji}$, $K_{ij}=K_{ji}$ and $\Lambda_{ij}=\Lambda_{ji}^*$. The induced symplectic transformation is defined by

$$\begin{pmatrix}\hat{\boldsymbol{q}}'\\ \hat{\boldsymbol{p}}'\end{pmatrix}\equiv U\begin{pmatrix}\hat{\boldsymbol{q}}\\ \hat{\boldsymbol{p}}\end{pmatrix}U^{-1}\equiv M\begin{pmatrix}\hat{\boldsymbol{q}}\\ \hat{\boldsymbol{p}}\end{pmatrix},\quad\text{or}\quad\begin{pmatrix}\boldsymbol{a}'\\ \boldsymbol{a}^{\dagger\prime}\end{pmatrix}\equiv U\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{a}^\dagger\end{pmatrix}U^{-1}\equiv T^{-1}MT\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{a}^\dagger\end{pmatrix}, \tag{2.2}$$

where M is a $2n\times2n$ real symplectic matrix and

$$T=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ -\mathrm{i}\mathbf{1} & \mathrm{i}\mathbf{1}\end{pmatrix}, \tag{2.3}$$

where $\mathbf{1}$ is the $n\times n$ unit matrix. The corresponding classical homogeneous linear canonical transformation is

$$\begin{pmatrix}\boldsymbol{q}'\\ \boldsymbol{p}'\end{pmatrix}=M^{-1}\begin{pmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{pmatrix},\quad\text{or}\quad\begin{pmatrix}\boldsymbol{\alpha}'\\ \boldsymbol{\alpha}^{*\prime}\end{pmatrix}=T^{-1}M^{-1}T\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\alpha}^*\end{pmatrix}. \tag{2.4}$$

The n-mode (canonical) coherent state with parameter $\boldsymbol{\alpha}$ is defined by

$$|\boldsymbol{\alpha}\rangle\equiv D(\boldsymbol{\alpha})|0\rangle=\exp(-\tfrac{1}{2}|\boldsymbol{\alpha}|^2)\exp(\boldsymbol{\alpha}\cdot\boldsymbol{a}^\dagger)|0\rangle\,, \tag{2.5}$$

where $|0\rangle$ is the n-mode ground state and $D(\boldsymbol{\alpha})\equiv\exp(\boldsymbol{\alpha}\cdot\boldsymbol{a}^\dagger-\boldsymbol{\alpha}^*\cdot\boldsymbol{a})$. It is easy to see that $D(\boldsymbol{\alpha})$ has the following properties:

$$D(\boldsymbol{\alpha})^\dagger=D(\boldsymbol{\alpha})^{-1}=D(-\boldsymbol{\alpha})\,, \tag{2.6}$$

$$UD(\boldsymbol{\alpha})U^{-1}=\exp(\boldsymbol{\alpha}\cdot\boldsymbol{a}^{\dagger\prime}-\boldsymbol{\alpha}^*\cdot\boldsymbol{a}')=\exp(\boldsymbol{\alpha}'\cdot\boldsymbol{a}^\dagger-\boldsymbol{\alpha}^{*\prime}\cdot\boldsymbol{a})=D(\boldsymbol{\alpha}')\,, \tag{2.7}$$

$$D(\boldsymbol{\alpha})\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{a}^\dagger\end{pmatrix}D(-\boldsymbol{\alpha})=\begin{pmatrix}\boldsymbol{a}-\boldsymbol{\alpha}\\ \boldsymbol{a}^\dagger-\boldsymbol{\alpha}^*\end{pmatrix}. \tag{2.8}$$

After these preliminaries, the main result of this paper can be expressed in the following theorem:

Theorem.

$$U=\lambda \int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\ |\boldsymbol{\alpha}'\rangle\langle\boldsymbol{\alpha}|, \qquad \lambda=[\pi^n(\langle 0|U|0\rangle)^*]^{-1}, \tag{2.9}$$

where

$$d^2(\boldsymbol{\alpha})=\prod_{k=1}^{n} d(\mathrm{Re}\,\alpha_k)\, d(\mathrm{Im}\,\alpha_k)=2^{-n}\prod_{k=1}^{n} dq_k\, dp_k .$$

This is a generalization of Fan's formula for one-mode and two-mode cases.

Proof. Let

$$V\equiv \int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\ |\boldsymbol{\alpha}'\rangle\langle\boldsymbol{\alpha}|=\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\, D(\boldsymbol{\alpha}')|0\rangle\langle 0|D(-\boldsymbol{\alpha}) . \tag{2.10}$$

$$U^{-1}V=\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\ U^{-1}D(\boldsymbol{\alpha}')|0\rangle\langle 0|D(-\boldsymbol{\alpha})=\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\ U^{-1}D(\boldsymbol{\alpha}')UU^{-1}|0\rangle\langle 0|D(-\boldsymbol{\alpha})$$

$$=\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\, D(\boldsymbol{\alpha})U^{-1}|0\rangle\langle 0|D(-\boldsymbol{\alpha}) . \tag{2.11}$$

Since $U^{-1}=\exp(-iH)$ with H spanned by $\{a_i^\dagger a_j^\dagger,\ a_i a_j,\ a_i^\dagger a_j+a_j a_i^\dagger\}$ which is a basis of the Lie algebra $sp(2n,\mathbb{C})$, according to the Baker–Campbell–Hausdorff formula, it is possible to factorize U^{-1} into

$$U^{-1}=\exp(\Gamma_{ij}a_i^\dagger a_j^\dagger)\exp[\Delta_{ij}(a_i^\dagger a_j+a_j a_i^\dagger)]\exp(E_{ij}a_i a_j) , \tag{2.12}$$

where Γ, Δ, E are complex-valued $n\times n$ matrices. Apply U^{-1} to $|0\rangle$ by using (2.12):

$$U^{-1}|0\rangle=\exp(\mathrm{Tr}\,\Delta)\exp(\Gamma_{ij}a_i^\dagger a_j^\dagger)|0\rangle , \tag{2.13}$$

and since $\exp(\mathrm{Tr}\,\Delta)=\langle 0|U^{-1}|0\rangle=\langle 0|U^\dagger|0\rangle=(\langle 0|U|0\rangle)^*$, we have

$$U^{-1}|0\rangle=(\langle 0|U|0\rangle)^*\exp(\Gamma_{ij}a_i^\dagger a_j^\dagger)|0\rangle . \tag{2.14}$$

Substitute (2.14) into (2.11):

$$U^{-1}V=(\langle 0|U|0\rangle)^*\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\, D(\boldsymbol{\alpha})\exp(\Gamma_{ij}a_i^\dagger a_j^\dagger)|0\rangle\langle 0|D(-\boldsymbol{\alpha})$$

$$=(\langle 0|U|0\rangle)^*\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\, D(\boldsymbol{\alpha})\exp(\Gamma_{ij}a_i^\dagger a_j^\dagger)D(-\boldsymbol{\alpha})D(\boldsymbol{\alpha})|0\rangle\langle 0|D(-\boldsymbol{\alpha})$$

$$=(\langle 0|U|0\rangle)^*\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\exp[\Gamma_{ij}(a_i^\dagger-\alpha_i^*)(a_j^\dagger-\alpha_j^*)]D(\boldsymbol{\alpha})|0\rangle\langle 0|D(-\boldsymbol{\alpha}) . \tag{2.15}$$

Using the identities

$$D(\boldsymbol{\alpha})|0\rangle\langle 0|D(-\boldsymbol{\alpha})=\exp(-\tfrac{1}{2}|\boldsymbol{\alpha}|^2)\exp(\boldsymbol{\alpha}\cdot\boldsymbol{a}^\dagger)|0\rangle\langle 0|\exp(\boldsymbol{\alpha}^*\cdot\boldsymbol{a})\exp(-\tfrac{1}{2}|\boldsymbol{\alpha}|^2)\,,$$

$$|0\rangle\langle 0| = :\exp(-\boldsymbol{a}^\dagger\cdot\boldsymbol{a}):\,,$$

then (2.15) becomes:

$$U^{-1}V=(\langle 0|U|0\rangle)^*\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\exp[\Gamma_{ij}(a_i^\dagger-\alpha_i^*)(a_j^\dagger-\alpha_j^*)]$$

$$\times\exp(-\tfrac{1}{2}|\boldsymbol{\alpha}|^2)\exp(\boldsymbol{\alpha}\cdot\boldsymbol{a}^\dagger):\exp(-\boldsymbol{a}^\dagger\cdot\boldsymbol{a}):\exp(\boldsymbol{\alpha}^*\cdot\boldsymbol{a})\exp(-\tfrac{1}{2}|\boldsymbol{\alpha}|^2)$$

$$=(\langle 0|U|0\rangle)^*:\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\exp[\Gamma_{ij}(a_i^\dagger-\alpha_i^*)(a_j^\dagger-\alpha_j^*)]\exp(-|\boldsymbol{\alpha}|^2+\boldsymbol{\alpha}\cdot\boldsymbol{a}^\dagger+\boldsymbol{\alpha}^*\cdot\boldsymbol{a}-\boldsymbol{a}^\dagger\cdot\boldsymbol{a}):\,. \tag{2.16}$$

Now the whole integration has been put inside the normal ordering symbol. According to the IWOP theory, we can perform the integration by treating $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ as "conjugate" c-numbers:

$$U^{-1}V=(\langle 0|U|0\rangle)^*:\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\exp[\Gamma_{ij}(\alpha_i^*-a_i^\dagger)(\alpha_j^*-a_j^\dagger)]\exp[-(\boldsymbol{\alpha}^*-\boldsymbol{a}^\dagger)\cdot(\boldsymbol{\alpha}-\boldsymbol{a})]:$$

$$=(\langle 0|U|0\rangle)^*:\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha}+\boldsymbol{a})\exp(\Gamma_{ij}\alpha_i^*\alpha_j^*)\exp(-|\boldsymbol{\alpha}|^2):$$

$$=(\langle 0|U|0\rangle)^*\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\exp(\Gamma_{ij}\alpha_i^*\alpha_j^*)\exp(-|\boldsymbol{\alpha}|^2)\,. \tag{2.17}$$

Since only the first term (which does not involve α_i^*) in the expansion of $\exp(\Gamma_{ij}\alpha_i^*\alpha_j^*)$ survives after integrating over $\boldsymbol{\alpha}$,

$$U^{-1}V=(\langle 0|U|0\rangle)^*\int_{-\infty}^{\infty} d^2(\boldsymbol{\alpha})\exp(-|\boldsymbol{\alpha}|^2)=\pi^n(\langle 0|U|0\rangle)^*=\lambda^{-1}\,. \tag{2.18}$$

Q.E.D.

3. Determination of the proportionality constant

From the theorem in the last section, we know that the proportionality constant $\lambda=[\pi^n(\langle 0|U|0\rangle)^*]^{-1}$ is a simple function of the vacuum expectation value $\langle 0|U|0\rangle$. Therefore, the determination of λ has been reduced to the problem of deriving an explicit expression of $\langle 0|U|0\rangle$ in terms of the coefficients of H. The derivation is as follows:

Define a t-dependent operator $U(t)\equiv\exp(\mathrm{i}tH)$, where $t\in[0, 1]$ is a parameter, so that $U=U(1)$. Since

$$\left[\mathrm{i}tH, \begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{a}^\dagger\end{pmatrix}\right]=\mathrm{i}t\begin{pmatrix}-\Lambda & -K\\ K^* & \Lambda^*\end{pmatrix}\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{a}^\dagger\end{pmatrix}, \tag{3.1}$$

we have

$$U(t)\begin{pmatrix} a \\ a^\dagger \end{pmatrix} U(t)^{-1} = \exp\left[it \begin{pmatrix} -A & -K \\ K^* & A^* \end{pmatrix} \right] \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \equiv \begin{pmatrix} A(t) & B(t) \\ B(t)^* & A(t)^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{3.2}$$

where $A(t)$ and $B(t)$ are $n\times n$ matrices whose elements are complex-valued functions of K_{ij}, K_{ij}^* *and* A_{ij} as well as t. It can be proved that [3]

$$\langle 0|U(t)|0\rangle = [\det A(t)]^{-1/2}, \tag{3.3}$$

$$0 < |\langle 0|U(t)|0\rangle| \leqslant 1, \tag{3.4}$$

which is a continuous complex-valued function in t. The branch of the square root is the one which makes $[\det A(t)]^{-1/2} \to +1$ as $t\to 0$. After the branch is determined, we can set $t=1$ and get a uniquely defined proportionality constant:

$$\lambda = [\pi^n (\langle 0|U|0\rangle)^*]^{-1} = \pi^{-n} [\det A(1)^*]^{1/2}, \tag{3.5}$$

an explicit expression in terms of K_{ij}, K_{ij}^* and A_{ij}.

4. Conclusion

The representation of U by integration over projection operators in the coherent-state basis having been proved for the general n-mode cases, we can derive the normally ordered form of n-mode U by applying IWOP to (2.9) as in the one-mode and two-mode cases. Then combined with (3.5), we will get the normally ordered form of U which is the same as those derived by other approaches [3,4].

Although (2.9) has here been applied to solve the normal ordering problem, because of its generality and explicit physical meaning it may have more applications in other mathematical physics problems.

The proof given here also explains why the IWOP technique only works for operators which induce *homogeneous* transformations. If U induces an inhomogeneous symplectic transformation, i.e., U is an exponential of an inhomogeneous quadratic form of operators, then (2.7) will not hold and we cannot have a relation similar to (2.11). In general there is no analogue of (2.9) for inhomogeneous cases.

Acknowledgement

I am grateful to Professor Y.S. Kim for bringing this problem to my attention and to Professor G.F. Chew and Professor E.H. Wichmann for their advice and patience. I would also like to thank Professor H. Fan for a stimulating discussion.

References

[1] H. Fan, H.R. Zaidi and J.R. Klauder, Phys. Rev. D 35 (1987) 1831;
H. Fan and J.R. Klauder, J. Phys. A 21 (1988) L725;
H. Fan and J. VanderLinde, Phys. Rev. A 39 (1989) 2987; 40 (1989) 4785; J. Phys. A 24 (1991) 2529.
[2] H. Fan, J. Phys. A 22 (1989) 3423; Phys. Rev. A 40 (1989) 4237; J. Math. Phys. 31 (1990) 257.
[3] F.A. Berezin, The method of second quantization (Academic Press, New York, 1966) p. 143.
[4] C.L. Mehta, J. Math. Phys. 18 (1977) 404;
G.P. Agrawal and C.L. Mehta, J. Math. Phys. 18 (1977) 408.