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Ermakov-Lewis invariant from the Wigner function of a squeezed coherent state

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Using the phase-space picture (Weyl-Wigner-Moyal formalism) of quantum mechanics for time-dependent Hamiltonians, we show that the Ermakov-Lewis invariant of a generalized harmonic oscillator can be derived from the Wigner function of a squeezed coherent state. The geometric meaning of this invariant is clarified and realized via the Wigner ellipse in phase space.

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I. INTRODUCTION

In the past three decades, there has been increasing interest in the Ermakov-Lewis (quantum) invariant [1–3] and its applications as well as generalizations [4–9]. For a harmonic oscillator of unit mass and time-dependent frequency with the Hamiltonian

$$\hat{H}(\hat{q},\hat{p};t) = \tfrac{1}{2}\{\hat{p}^2 + [\omega(t)]^2\hat{q}^2\}, \tag{1}$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators, the Ermakov-Lewis invariant $\hat{I}=\hat{I}(t)$ is defined as

$$\hat{I} = \frac{1}{2}\left(\lambda\frac{\hat{q}^2}{\theta^2} + (\theta\hat{p} - \dot{\theta}\hat{q})^2\right), \qquad \dot{\theta} \equiv \frac{d\theta}{dt}, \tag{2}$$

where $\theta=\theta(t)$ is any (real or complex) solution of the nonlinear differential equation

$$\ddot{\theta} + [\omega(t)]^2\theta = \frac{\lambda}{\theta^3}, \tag{3}$$

with λ being a constant. $\hat{I}$ is a quantum invariant in the sense that

$$\frac{d\hat{I}}{dt} = \frac{\partial\hat{I}}{\partial t} + i[\hat{H},\hat{I}] = 0 \qquad (\hbar = 1). \tag{4}$$

The above definition of the Ermakov-Lewis invariant can be generalized for a time-dependent "generalized harmonic oscillator" with the Hamiltonian

$$\hat{H}(\hat{q},\hat{p};t) = \tfrac{1}{2}\left[a(t)\hat{q}^2 + b(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) + c(t)\hat{p}^2\right] + f(t)\hat{q} + g(t)\hat{p}, \tag{5}$$

where

$$a(t), c(t) > 0 \text{ and } a(t)c(t) - [b(t)]^2 > 0 \text{ for all } t. \tag{6}$$

Since (5) is inhomogeneously quadratic in $\hat{q}$ and $\hat{p}$, in principle we can study the quantum problems associated with this Hamiltonian in the phase-space picture (Weyl-Wigner-Moyal formalism) via the Wigner functions. So long as the Hamiltonian of a quantum system is (inhomogeneously) quadratic, the equation of motion of the Wigner function is of the same form as the classical Liouville equation. Hence time evolution of the Wigner function can be obtained directly from the solution of the equation of motion of the corresponding classical system. Although the phase-space picture is usually used for time-independent Hamiltonians [10–13], we will show in Sec. II that it is also valid when the Hamiltonian is time-dependent.

The purpose of this paper is to use the phase-space approach to (i) study the relations between the Ermakov-Lewis invariant for the Hamiltonian (5) and the Wigner function of a squeezed coherent state, (ii) derive the Ermakov-Lewis invariant from the relations found in (i), and (iii) give geometric meaning to this invariant in the phase-space picture via the Wigner ellipse.

II. PHASE-SPACE PICTURE FOR TIME-DEPENDENT HAMILTONIANS

For a quantum system described by the Hamiltonian $\hat{H}=\hat{H}(\hat{q},\hat{p};t)$, the density operator $\hat{\rho}$ satisfies the von Neumann–Landau equation

$$\frac{d\hat{\rho}}{dt} = \frac{\partial\hat{\rho}}{\partial t} + i[\hat{H},\hat{\rho}] = 0. \tag{7}$$

This equation is valid for both time-dependent and time-independent $\hat{H}$. Since the equation of motion of the Wigner function can be derived from (7) [11, 12], it is of the same form for both time-dependent and time-independent Hamiltonians.

The above discussion is true for any Hamiltonian. Now we restrict the Hamiltonian to be (5) which is inhomogeneously quadratic. As in the time-independent case, we still have the classical Liouville equation as the equation of motion for the Wigner function:

$$\frac{\partial}{\partial t}W(x,k;t) + \frac{\partial H}{\partial k}\frac{\partial}{\partial x}W(x,k;t) - \frac{\partial H}{\partial x}\frac{\partial}{\partial k}W(x,k;t) = 0, \tag{8}$$

where $H=H(x,k;t)$ is the classical correspondent of (5) by the Weyl correspondence rule (the symmetrization rule):

$$H(x,k;t) = \tfrac{1}{2}\left[a(t)x^2 + 2b(t)xk + c(t)k^2\right] + f(t)x + g(t)k, \tag{9}$$

with x and k being the canonical coordinate and momentum.

Using Hamilton's canonical equations in classical mechanics

$$\dot{x} = \frac{\partial H}{\partial k}, \qquad \dot{k} = -\frac{\partial H}{\partial x}, \tag{10}$$

(8) can be rewritten as

$$\left(\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{k}\frac{\partial}{\partial k}\right) W(x,k;t) = \frac{d}{dt}W(x,k;t) = 0. \tag{11}$$

Introducing the notation $\mathbf{z}=\binom{x}{k}$, Hamilton's equations (10) with H defined in (9) become

$$\dot{\mathbf{z}} = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}\mathbf{z} + \begin{pmatrix} g \\ -f \end{pmatrix}, \tag{12}$$

and the solution can be denoted by

$$\mathbf{z}(t) = R(t)\mathbf{z}(0) + \boldsymbol{\zeta}(t), \quad R(0) = \mathbf{1} \text{ and } \boldsymbol{\zeta}(0) = \mathbf{0}, \tag{13}$$

where $R(t)\mathbf{z}(0)$ is the solution of the corresponding homogeneous equation

$$\dot{\mathbf{z}} = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}\mathbf{z}, \tag{14}$$

and $\boldsymbol{\zeta}(t)$ is a special solution of (12). The geometric meaning of $\boldsymbol{\zeta}(t)$ is the trajectory traced by the point which is initially at the origin in phase space. The solution (13) is essentially a time-dependent inhomogeneous linear transformation and hereafter we will call it "phase flow."

From (13), the general solution of (11) can be obtained as

$$W(x,k;t) = W(\mathbf{z};t) = W(R^{-1}(t)\left[\mathbf{z} - \boldsymbol{\zeta}(t)\right]; t = 0), \tag{15}$$

or equivalently,

$$W(\mathbf{z}(t);t) = W(\mathbf{z}(0);0). \tag{16}$$

Hence time evolution of the Wigner function follows the phase flow (13) exactly.

III. SQUEEZED COHERENT STATES IN PHASE SPACE

In the phase-space picture, quantum states are represented by the Wigner functions [10–13]. For the most general one-mode squeezed coherent state [14–16], the Wigner function at a fixed time, say t=0, is

$$W(\mathbf{z};0) = \frac{\sqrt{\Delta}}{\pi}\exp\left[-(\mathbf{z}-\mathbf{z}_0)^\top M(\mathbf{z}-\mathbf{z}_0)\right], \tag{17}$$

where

$$M = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}, \quad \alpha,\beta > 0, \quad 0 < \Delta = \det M \le 1, \tag{18}$$

and $\top$ denotes the transpose of a matrix. Formula (17) represents a zero-temperature (pure) squeezed coherent state when Δ=1; otherwise, it corresponds to a thermal squeezed coherent state. The ground state, thermal state, coherent state, and squeezed state are all special cases of (17). There is no physical state corresponding to $\Delta > 1$ due to the constraint of the uncertainty relation.

The physical meaning of the vector $\mathbf{z}_0$ is $\mathbf{z}_0 = \binom{\langle\hat{q}\rangle}{\langle\hat{p}\rangle}$, where the expectation values are with respect to the state represented by (17). The matrix M is related to the covariance matrix Σ in the following way:

$$\Sigma = \begin{pmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{qp} & \sigma_{pp} \end{pmatrix} = \tfrac{1}{2}M^{-1}, \tag{19}$$

where

$$\sigma_{qq} = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2, \tag{20}$$

$$\sigma_{pp} = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2, \tag{21}$$

and

$$\sigma_{qp} = \left\langle \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \right\rangle - \langle\hat{q}\rangle\langle\hat{p}\rangle. \tag{22}$$

Therefore, we can express the matrix M in terms of the elements in Σ as

$$M = \frac{1}{2\det(\Sigma)}\begin{pmatrix} \sigma_{pp} & -\sigma_{qp} \\ -\sigma_{qp} & \sigma_{qq} \end{pmatrix} = 2\Delta\begin{pmatrix} \sigma_{pp} & -\sigma_{qp} \\ -\sigma_{qp} & \sigma_{qq} \end{pmatrix}. \tag{23}$$

Since (17) is a Gaussian distribution in the canonical variables $\mathbf{z}$, we can use a contour for $W(\mathbf{z};0)$=const in phase space as the geometric representation of this

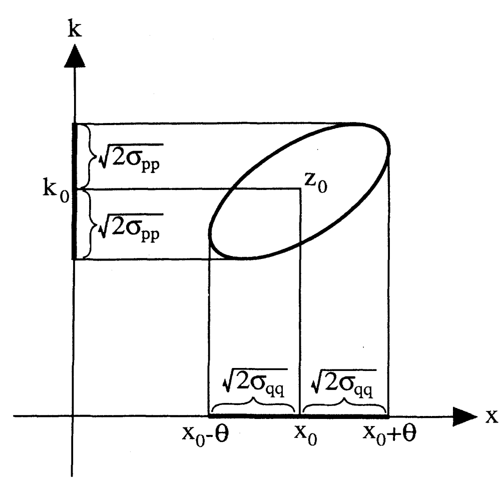

FIG. 1. Wigner ellipse of a squeezed coherent state in phase space.

Wigner function. Conventionally, we choose the constant to be $\sqrt{\Delta}/(\pi e)$ such that the corresponding contour is

$$(\mathbf{z}-\mathbf{z}_0)^\top M(\mathbf{z}-\mathbf{z}_0) = 1. \tag{24}$$

We will call this contour the "Wigner ellipse." It is in general an ellipse centered at $\mathbf{z}_0$ with its shape determined by the matrix M. Equations (24) and (17) are equivalent since a Gaussian distribution is completely determined by its first and second moments.

The projection of the Wigner ellipse onto the x axis represents $\hat{q}$ uncertainty with respect to the state represented by the ellipse. The length of this projection is $2\sqrt{\beta/\Delta}=2\sqrt{2\sigma_{qq}}$, while the projection onto the k axis represents $\hat{p}$ uncertainty and is of length $2\sqrt{\alpha/\Delta}=2\sqrt{2\sigma_{pp}}$ (Fig. 1).

IV. TIME EVOLUTION OF SQUEEZED COHERENT STATES

For the Hamiltonian (5), according to (15) time evolution of the Wigner function (17) is

$$W(\mathbf{z};t) = \frac{\sqrt{\Delta}}{\pi}\exp\{-[\mathbf{z}-\mathbf{z}_0(t)]^\top M(t)\,[\mathbf{z}-\mathbf{z}_0(t)]\}, \tag{25}$$

where

$$\mathbf{z}_0(t) = R(t)\mathbf{z}_0 + \boldsymbol{\zeta}(t), \tag{26}$$

and

$$M(t) = [R(t)]^{-\top} M [R(t)]^{-1}, \quad [R(t)]^{-\top} \equiv ([R(t)]^{-1})^\top. \tag{27}$$

Equation (26), which is the time evolution of the canonical conjugate variables x_0 and k_0, is essentially a canonical transformation, hence

$$\frac{\partial(x_0(t),k_0(t))}{\partial\,(x_0,k_0)} = \det R(t) = 1, \tag{28}$$

and

$$\det M(t) = \det M = \Delta. \tag{29}$$

From (25), time evolution of the corresponding Wigner ellipse is

$$[\mathbf{z}-\mathbf{z}_0(t)]^\top M(t)\,[\mathbf{z}-\mathbf{z}_0(t)] = 1. \tag{30}$$

Equation (29) guarantees that (30) remains an ellipse with constant area $\pi/\sqrt{\Delta}$ at all times. According to (26) and (27), both the center and each point on the boundary of the ellipse (30) follow the phase flow. Notice that $M(t)$ is independent of the center, hence the evolution of the center and the shape can be studied separately. Moreover, since $M(t)$ only depends on $R(t)$, which constitutes the solution of the homogeneous equation (14), time evolution of the shape is determined entirely by (14) and is independent of the linear terms in the Hamiltonian (5).

Corresponding to (27), time evolution of the covariance matrix is

$$\Sigma(t) = R(t)\Sigma[R(t)]^\top, \tag{31}$$

with

$$\det\Sigma(t) = \det\Sigma = \frac{1}{4\Delta}. \tag{32}$$

The following two relations among $\sigma_{qq}(t)$, $\sigma_{pp}(t)$, and $\sigma_{qp}(t)$ are easy to prove using the Heisenberg picture:

$$c(t)\sigma_{qp}(t) = \tfrac{1}{2}\dot{\sigma}_{qq}(t) - b(t)\sigma_{qq}(t), \tag{33}$$

$$c(t)\sigma_{pp}(t) = a(t)\sigma_{qq}(t) + \dot{\sigma}_{qp}(t), \tag{34}$$

where the Heisenberg equation

$$\frac{d}{dt}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} = \begin{pmatrix} b & c\\ -a & -b\end{pmatrix}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} + \begin{pmatrix} g\\ -f\end{pmatrix} \tag{35}$$

has been used. Actually, since we already know that time evolution of $M(t)$, hence $\Sigma(t)$, is independent of the linear terms in the Hamiltonian (5), we may only use the homogeneous part of (5) to derive the above relations.

According to (33) and (34), both $\sigma_{pp}(t)$ and $\sigma_{qp}(t)$ can be expressed in terms of $\sigma_{qq}(t)$ and its derivatives; therefore, $\Sigma(t)$ is determined *solely* by $\sigma_{qq}(t)$ [17]. From the constraint (32), $\sigma_{qq}(t)$ must satisfy the nonlinear differential equation

$$2\sigma_{qq}\ddot{\sigma}_{qq} - \dot{\sigma}_{qq}^2 + 2N\sigma_{qq}\dot{\sigma}_{qq} + 4K\sigma_{qq}^2 = \frac{c^2}{\Delta}, \tag{36}$$

where

$$N = -\frac{\dot{c}}{c},$$
$$K = ac - b^2 - \dot{b} - bN = ac - b^2 - c\frac{d}{dt}\left(\frac{b}{c}\right). \tag{37}$$

Since $\sigma_{qq}(t) > 0$, we may make a change of variable

$$\sigma_{qq}(t) = \tfrac{1}{2}[\theta(t)]^2, \quad \theta(t) > 0, \tag{38}$$

then (36) will be transformed into a simpler form

$$\ddot{\theta} + N\dot{\theta} + K\theta = \frac{c^2}{\Delta\theta^3}. \tag{39}$$

The geometric meaning of $\theta(t)$ is that $[x_0(t) \pm \theta(t)]$ are the coordinates of two end points of the x projection of the Wigner ellipse (Fig. 1).

In terms of $\theta(t)$, $\Sigma(t)$ can be rewritten as

$$\Sigma(t) = \Sigma(\theta(t)) = \frac{1}{2}\begin{bmatrix}\theta^2 & \theta L(\theta)\\ \theta L(\theta) & \frac{1}{\Delta\theta^2} + [L(\theta)]^2\end{bmatrix}, \tag{40}$$

where

$$L(\theta) \equiv \frac{\dot{\theta} - b\theta}{c}. \tag{41}$$

Correspondingly, the matrix $M(t)$ becomes

$$M(t)=M(\theta(t))=\Delta\begin{bmatrix}\frac{1}{\Delta\theta^2}+[L(\theta)]^2 & -\theta L(\theta)\\ -\theta L(\theta) & \theta^2\end{bmatrix}. \tag{42}$$

Comparing the $(1,1)$ elements in (40) and (31), we see that the solution $\theta(t)$ of the nonlinear differential equation (39) can be obtained from $R(t)$, which constitutes the solutions of (12) and (14). This is an example of the well-known nonlinear superposition law. There are two independent solutions of (39) corresponding to three parameters in Σ with one constraint $\det\Sigma=\frac{1}{4\Delta}$.

V. ERMAKOV-LEWIS INVARIANT FROM A SQUEEZED COHERENT STATE

Comparing the von Neumann–Landau equation (7) with the definition of a quantum invariant (4), it is obvious that the density operator $\hat{\rho}$ is a special kind of quantum invariant. Since the Wigner function is equivalent to the density operator [18], it can be taken as a "c-number quantum invariant."

Because the Hamiltonian (5) is inhomogeneously quadratic, for a quantum invariant $\hat{I}$ which satisfies (4), by analogy to (11) the c-number quantum invariant I_W corresponding to this $\hat{I}$ must satisfy

$$\left(\frac{\partial}{\partial t}+\dot{x}\frac{\partial}{\partial x}+\dot{k}\frac{\partial}{\partial k}\right)I_W(x,k;t)=\frac{d}{dt}I_W(x,k;t)=0, \tag{43}$$

and vice versa. The equation of motion for the Wigner function (11) is a special case of the above equation.

For the Wigner function (25) that corresponds to a general squeezed coherent state, the equation of motion (11) is mathematically equivalent to

$$\frac{d}{dt}\left([\mathbf{z}-\mathbf{z}_0(t)]^\top M(t)\,[\mathbf{z}-\mathbf{z}_0(t)]\right)=0. \tag{44}$$

Comparing the above equation with (43), we find that

$$[\mathbf{z}-\mathbf{z}_0(t)]^\top M(t)\,[\mathbf{z}-\mathbf{z}_0(t)] \tag{45}$$

is also a c-number quantum invariant, where $\mathbf{z}_0(t)$ and $M(t)$ are defined by (26) and (42), respectively. According to the Weyl correspondence rule, the operator corresponding to (45) is

$$\left[\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}-\mathbf{z}_0(t)\right]^\top M(t)\left[\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}-\mathbf{z}_0(t)\right]. \tag{46}$$

Dividing this operator by 2Δ, we get the Ermakov-Lewis invariant with restricted parameters for the generalized harmonic oscillator (5):

$$\hat{I}=\frac{1}{2\Delta}\left[\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}-\mathbf{z}_0(t)\right]^\top M(t)\left[\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}-\mathbf{z}_0(t)\right] \tag{47}$$

$$=\frac{1}{2}\left(\frac{[\hat{q}-x_0(t)]^2}{\Delta\theta^2}+\{\theta(\hat{p}-k_0(t))-L(\theta)[\hat{q}-x_0(t)]\}^2\right). \tag{48}$$

Since the classical correspondent (the Weyl symbol) of the Ermakov-Lewis invariant (48) is proportional to the exponent of the Wigner function of a squeezed coherent state, the geometric realization of this invariant in phase space is nothing but the Wigner ellipse (30).

Now we can release the constraints (18) on the matrix M and extend α, β, and γ as well as $\mathbf{z}_0$ to arbitrary complex numbers. After this extension, $\hat{I}$ in (48) is an invariant if and only if $\mathbf{z}_0(t)$ and $\theta(t)$ are arbitrary solutions of (12) and (39), respectively. Hence (48) becomes the Ermakov-Lewis invariant for the generalized harmonic oscillator (5). *From now on all of the related formulas will be with respect to this extension.* Of course, there is no correspondence of this extension in phase space.

For a fixed Δ, (48) has two degrees of freedom which correspond to the two independent solutions of (39). Therefore there are exactly two independent Ermakov-Lewis invariants of the form (48).

Although the Ermakov-Lewis invariant is often expressed in the form of (48), the physical interpretation for it being an invariant is easier to understand via (47). Using the Heisenberg picture, the Ermakov-Lewis invariant $\hat{I}$ at a fixed time t can be denoted as

$$\hat{I}(t)=\frac{1}{2\Delta}\left[\begin{pmatrix}\hat{q}(t)\\ \hat{p}(t)\end{pmatrix}-\mathbf{z}_0(t)\right]^\top M(t)\left[\begin{pmatrix}\hat{q}(t)\\ \hat{p}(t)\end{pmatrix}-\mathbf{z}_0(t)\right]. \tag{49}$$

Since the Heisenberg equation (35) is linear and isomorphic to (12), the time evolution of $\hat{q}(t)$ and $\hat{p}(t)$ is analogous to (13):

$$\begin{pmatrix}\hat{q}(t)\\ \hat{p}(t)\end{pmatrix}=R(t)\begin{pmatrix}\hat{q}(0)\\ \hat{p}(0)\end{pmatrix}+\boldsymbol{\zeta}(t). \tag{50}$$

Substitute the above expression and the definitions of $\mathbf{z}_0(t)$ in (26) and $M(t)$ in (27) into (49), and we find that $\hat{I}(t)$ is identical to

$$\hat{I}(0)=\frac{1}{2\Delta}\left[\begin{pmatrix}\hat{q}(0)\\ \hat{p}(0)\end{pmatrix}-\mathbf{z}_0\right]^\top M\left[\begin{pmatrix}\hat{q}(0)\\ \hat{p}(0)\end{pmatrix}-\mathbf{z}_0\right]. \tag{51}$$

It is now clear that $\hat{I}$ is an invariant because the time evolution of the operators and that of the c-numbers in $\hat{I}$

always cancel each other. Hence $\hat{I}$ is an invariant in the sense that its expectation value with respect to a given state is a constant and equal to that of $\hat{I}(0)$ at all times.

For homogeneous Hamiltonians, i.e., (5) with $f(t)=g(t)=0$, since the initial conditions of the Wigner ellipse are arbitrary, we can set $\mathbf{z}_0=0$ so that $\mathbf{z}_0(t)=0$ for all t. This will guarantee that the Ermakov-Lewis invariant is a homogeneous quadratic form [5]:

$$\hat{I} = \frac{1}{2}\left(\frac{\hat{q}^2}{\Delta\theta^2} + \left[\theta\hat{p} - L(\theta)\hat{q}\right]^2\right). \tag{52}$$

Therefore we can always have a homogeneous quadratic Ermakov-Lewis invariant so long as the Hamiltonian is homogeneous. The original Ermakov-Lewis invariant (2) can be obtained from (52) by taking $a(t)=[\omega(t)]^2$, $b(t)=0$, $c(t)=1$, and setting Δ equal to λ^{-1}.

The above discussion is not true when the Hamiltonian is inhomogeneous, where the Ermakov-Lewis invariant (48) must be a linear combination of the quadratic, the linear, and the zero-degree terms.

VI. CALCULATION OF THE ERMAKOV-LEWIS INVARIANT

The Ermakov-Lewis invariant derived in the last section is determined by the solutions of both Hamilton's equation (12) and the nonlinear differential equation (39). However, according to the nonlinear superposition law discussed in Sec. IV, the general solutions of (39) can be obtained from $R(t)$, which constitutes the solution of (12). Therefore the Ermakov-Lewis invariant is determined solely by (12).

In order to solve (12), we first decouple this system of equations into

$$\ddot{x} + N\dot{x} + Kx = F, \tag{53}$$

and

$$k = \frac{1}{c}(\dot{x} - bx - g) = L(x) - \frac{g}{c}, \tag{54}$$

where $F = bg - cf + \dot{g} + gN$, and N and K are defined as in (37). This decoupling reduces the system of equations (12) into a single equation (53). Notice that (53) and (39) differ only by the term on the right-hand side. This is not a coincidence but has a geometric meaning that manifests itself when $\theta(t)\gg 1$ for a highly squeezed Wigner ellipse.

Equation (53) can be put into a simpler form by using the transformation

$$y = x\exp\left(\tfrac{1}{2}\int_{t_0}^{t} N(\tau)d\tau\right), \tag{55}$$

the new differential equation taking the form

$$\ddot{y} + Qy = G, \tag{56}$$

with

$$Q = K - \tfrac{1}{2}\dot{N} - \tfrac{1}{4}N^2, \tag{57}$$

and

$$G = F\exp\left(\tfrac{1}{2}\int_{t_0}^{t} N(\tau)d\tau\right). \tag{58}$$

From the theory of ordinary differential equations, we know that the general solutions of the inhomogeneous equation (56) can be constructed directly from the solutions of the corresponding homogeneous equation

$$\ddot{y} + Qy = 0. \tag{59}$$

Therefore in order to calculate the Ermakov-Lewis invariant for the Hamiltonian (5), the only equation we really have to solve is the homogeneous differential equation (59). Although the exact solutions are rare, the numerical solution of this equation is easy to obtain for any $Q=Q(t)$.

VII. DEGENERATE CASE: LINEAR INVARIANTS

In Sec. V we derived the Ermakov-Lewis invariant (48) for a generalized harmonic oscillator (5), which is in general inhomogeneously quadratic in $\hat{q}$ and $\hat{p}$. We shall show in this section that (48) will also give all possible linear invariants in the limit $\Delta \to \infty$.

According to (42), the matrix M/Δ is a well-defined finite matrix when $\Delta \to \infty$. In this limit, (48) degenerates into

$$\hat{I} = \tfrac{1}{2}\{\phi(\hat{p} - k_0(t)) - L(\phi)\left[\hat{q} - x_0(t)\right]\}^2, \tag{60}$$

where $\mathbf{z}_0(t)=\binom{x_0(t)}{k_0(t)}$ is still an arbitrary solution of (12), while ϕ is an arbitrary solution of the following equation:

$$\ddot{\phi} + N\dot{\phi} + K\phi = 0, \tag{61}$$

which is the corresponding homogeneous equation of (39) and (53). Therefore we can apply the technique of solving (53) discussed in the last section to solve (61).

Taking the "square root" of the degenerate Ermakov-Lewis invariant (60), we get the linear quantum invariant

$$\hat{J} = \frac{1}{\sqrt{2}}\{\phi(\hat{p} - k_0(t)) - L(\phi)\left[\hat{q} - x_0(t)\right]\}\ . \tag{62}$$

Since there are two independent solutions for (61), $\hat{J}$ in (62) has two degrees of freedom and hence corresponds to two independent linear invariants [9, 19].

Analogously to the nonlinear superposition law discussed in Sec. IV, the two independent solutions of (39) can be obtained from those of (61). This means that in principle the quadratic Ermakov-Lewis invariant $\hat{I}$ in (48) can be constructed from the linear invariant $\hat{J}$ in (62).

VIII. DISCUSSION

In this paper, we introduce a method for deriving the Ermakov-Lewis invariant for a time-dependent generalized harmonic oscillator via the Wigner function of a squeezed coherent state. The derivation and calculation are simplified and the geometric meaning appears automatically.

In [7], the authors discussed several methods to derive the Ermakov-Lewis invariant for an inhomogeneous Hamiltonian which is our (5) with $c(t)=1$ and $b(t)=g(t)=0$. Their conclusion is that there must be one more auxiliary equation in order to include the linear term in the Hamiltonian. It is obvious from our derivation that the "extra" auxiliary equation is nothing but the classical equation of motion (53) and there is no essential difference between the homogeneous case and inhomogeneous case.

The geometric as well as physical meanings of the Ermakov-Lewis invariant have been discussed in the literature by invoking an auxiliary plane [2] or a complex plane [8], etc. In the phase-space picture, we associate the Ermakov-Lewis invariant with the Wigner ellipse defined in phase space without introducing another kind of plane.

Since $\hat{q}$ and $\hat{p}$ are on the same footing in the phase-space picture, we can also use

$$\sigma_{pp}(t) = \tfrac{1}{2}[\eta(t)]^2, \quad \eta(t) > 0, \tag{63}$$

to rederive all the above results.

Because the (inhomogeneous) quadratic invariants in terms of creation and annihilation operators [19–23] are essentially equivalent to the Ermakov-Lewis invariant, these invariants also can be constructed and studied by the approach introduced in this paper.

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[17] This conclusion might seem strange at first sight because for a fixed Δ, i.e., a fixed area, there are *two* independent degrees of freedom for a Wigner ellipse at any time t. Intuitively, these two degrees of freedom can be understood as corresponding to the two initial conditions of the second-order differential equation (36) defined below.
[18] More precisely, the Wigner function is proportional to the classical correspondent (the Weyl symbol) of the density operator according to the Weyl correspondence rule, viz., Weyl symbol of $\hat{\rho}=2\pi W(x,k)$.
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