E(2)-symmetric two-mode sheared states

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It is noted that the symmetry of two-mode squeezed states of light is governed by the group Sp(4) that is locally isomorphic to O(3,2). This group has subgroups that are locally isomorphic to the two-dimensional Euclidean group. Two-mode states having the E(2) symmetry are constructed. The translation-like transformations of this symmetry group shear the Wigner distribution function defined over the four-dimensional phase space consisting of two pairs of canonical variables. Sheared states are constructed in the Schrödinger picture of quantum mechanics and in the Fock space for photon numbers. It is shown that the Wigner phase-space picture is a convenient representation of quantum mechanics for calculating measurable quantities of the sheared states.

I. INTRODUCTION

The basic symmetry of the coherent state of light is that of the Heisenberg group h(4).[1,2] The symmetry of the one-mode squeezed state is dictated by the group Sp(2), which is locally isomorphic to O(2,1).[3] For two-mode squeezed states, the SU(2) and SU(1,1) symmetries have been discussed extensively in the literature.[3–5] It has also been noted that the basic symmetry for the two-mode state is that of the group Sp(4), which is locally isomorphic to the $(3+2)$-dimensional deSitter group.[3,6,7] The group Sp(4) has both SU(2)-like and SU(1,1)-like subgroups. We use the word "like" to indicate that the correspondence is locally isomorphic.

While both SU(2) and SU(1,1) are three-parameter groups, there are three-parameter subgroups of Sp(4) that are locally isomorphic to the two-dimensional Euclidean group, which is commonly denoted by E(2).[8,9] The group E(2) is locally isomorphic to the internal space-time symmetry group of massless particles.[3,10,11] It is also widely known that this group can be obtained from the three-dimensional rotation group or from the group O(2,1) through group contraction.[12,13]

The word E(2) is also associated with one-mode coherent state in the Wigner phase-space picture of quantum mechanics.[3,14] However, in the Schrödinger picture, the symmetry of coherent state is that of h(4), which is known as the Heisenberg group.[1] Indeed, the coherent state exhibits different symmetry properties in two different representations of quantum mechanics.

In this paper, we shall study the representation having the E(2)-like symmetry in both the Schrödinger and Wigner pictures by considering the E(2)-like subgroups of O(3,2). As Dirac noted in 1963,[6] it is possible to construct two-oscillator representations of O(3,2), which later became the basic language for two-mode squeezed states of light.[7] If we translate this symmetry to the Wigner function of two canonical variables, it becomes that of Sp(4), as the group Sp(4) is locally isomorphic to O(3,2). The SU(2)-like and SU(1,1)-like subgroups of Sp(4) have been thoroughly studied in the literature.[3–5,15] In this paper, we shall construct a generalized coherent state having the E(2)-like symmetry in both the Schrödinger and Wigner pictures. In so doing, we shall note first that the E(2)-like transformations shear the Wigner function in four-dimensional phase space leading to a sheared state.

In Sec. II, we formulate the problem by introducing the symmetry group of the two-mode Wigner function defined in four-dimensional phase space. It is noted that the symmetry is that of Sp(4). In Sec. III, the three-parameter subgroups of Sp(4) are discussed in detail. It is shown that, in addition to SU(2) and SU(1,1), there are subgroups that are locally isomorphic to the two-dimensional Euclidean group.

In Sec. IV, we explain in detail how the E(2)-like transformations lead to shears in the four-dimensional Wigner phase space. It is noted that there are shears that can be separated between the modes and those that are coupled. All coupled shears can be uncoupled through a rotation in the E(2)-like symmetry group. In Sec. V, the E(2)-like symmetry is translated into the symmetry operations applicable to harmonic oscillator wave functions in the Schrödinger picture. In Sec. VI, we study how the symmetry operation affects the photon distribution in the occupation number space commonly known as Fock space. Finally, in Sec. VII, it is shown that the Wigner distribution function is very convenient for extracting measurable quantities from the sheared states.

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II. FOUR-DIMENSIONAL PHASE SPACE FOR TWO-MODE STATES

The Wigner phase-space picture of quantum mechanics is the natural language for squeezed states of light.[3,10] The Wigner distribution function for two-mode states is defined over the four-dimensional phase space consisting of two pairs of canonical variables x_1, p_1 and x_2, p_2.[3,7] The group of homogeneous linear transformations in this phase space is Sp(4).[7]

If M is a 4×4 matrix representing homogeneous linear canonical transformations applicable to the four-dimensional vector space of (x_1,x_2,p_1,p_2), it satisfies the condition

$$M \quad J \quad \tilde{M}=J, \tag{2.1}$$

where

$$J=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

From this condition, it is possible to find the generators of the group. In terms of the generator G, M can be written as

$$M=e^{-i\alpha G}, \tag{2.2}$$

where G represents a set of purely imaginary 4×4 matrices that commute with J while antisymmetric or anticommute with J while symmetric.[7]

There are ten linearly independent generators. In terms of the Pauli matrices, they are[7,9,16]

$$J_1=\frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad J_2=\frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix},$$
$$J_3=\frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad J_0=\frac{i}{2}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.3}$$

and

$$K_1=-\frac{i}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad K_2=\frac{i}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix},$$
$$K_3=\frac{i}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$
$$Q_1=\frac{i}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \quad Q_2=-\frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix},$$
$$Q_3=\frac{i}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.4}$$

These generators satisfy the commutation relations:

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \quad [J_i,J_0]=0,$$
$$[J_i,K_j]=i\epsilon_{ijk}K_k, \quad [J_i,Q_j]=i\epsilon_{ijk}Q_k,$$
$$[K_i,K_j]=[Q_i,Q_j]=-i\epsilon_{ijk}J_k, \tag{2.5}$$
$$[K_i,Q_j]=i\delta_{ij}J_0,$$
$$[K_i,J_0]=iQ_i, \quad [Q_i,J_0]=-iK_i.$$

Indeed, these are the generators of the group Sp(4), which is locally isomorphic to the $(3+2)$-dimensional deSitter group.[7,9]

Let us now introduce the Wigner function. If $\psi(x)$ is a wave function in the Schrödinger picture of quantum mechanics, then the Wigner phase-space distribution function for one pair of canonical variables is[3,7,17,18]

$$W(x,p)=\frac{1}{\pi}\int e^{2ipy}\psi^*(x+y)\psi(x-y)dy. \tag{2.6}$$

The parameters x and p are c numbers. The Wigner function is defined over the two-dimensional phase space of x and p. For studying coherent and squeezed states of light, we can start with the ground-state harmonic oscillator with the Schrödinger wave function[19]

$$\psi(x)=(1/\pi)^{1/4}e^{-x^2/2}. \tag{2.7}$$

The Wigner function for this ground state is

$$W(x,p)=(1/\pi)\exp\{-(x^2+p^2)\}. \tag{2.8}$$

We can obtain all known coherent and squeezed states by making canonical transformations of the above expression.[19] The group of linear canonical transformations consists of translations, rotations, and squeezes.

If the wave function depends on two coordinate variables x_1 and x_2, the Wigner function will depend on two pairs of canonical variables:

$$W(x_1,x_2,p_1,p_2)=\left(\frac{1}{\pi}\right)^2\int \exp\{2i(p_1y_1+p_2y_2)\}$$
$$\times\psi^*(x_1+y_1,x_2+y_2)$$
$$\times\psi(x_1-y_1,x_2-y_2)dy_1\,dy_2. \tag{2.9}$$

Let us then consider the ground-state wave function for this two-oscillator system:

$$\psi(x_1,x_2)=(1/\pi)^{1/2}\exp\{-(\tfrac{1}{2})(x_1^2+x_2^2)\}. \tag{2.10}$$

Then the Wigner function will be

$$W(x_1,x_2,p_1,p_2)=(1/\pi)^2\exp\{-(x_1^2+x_2^2+p_1^2+p_2^2)\}. \tag{2.11}$$

We are now interested in performing rotations and squeezes with respect to two pairs of variables. There are three possible ways of choosing two pairs among the four variables, and the above Wigner function can be written in three different ways:[7]

$$\begin{aligned} W(x_1,x_2,p_1,p_2) &= (1/\pi)^2\exp\{-(x_1^2+p_1^2)\}\exp\{-(x_2^2+p_2^2)\} \\ &= (1/\pi)^2\exp\{-(x_1^2+p_2^2)\}\exp\{-(x_2^2+p_1^2)\} \\ &= (1/\pi)^2\exp\{-(x_1^2+x_2^2)\}\exp\{-(p_1^2+p_2^2)\}. \end{aligned} \tag{2.12}$$

For this Wigner function, the generators of Sp(4) take the differential forms.[7] As in the case Eq. (2.3), there are four generators of rotations:

$$\begin{aligned} J_1 &= +\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial x_1}\right)-\left(x_2\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial x_2}\right)\right\}, \\ J_2 &= +\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial x_1}\right)+\left(x_2\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial x_2}\right)\right\}, \\ J_3 &= -\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right)+\left(p_1\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial p_1}\right)\right\}, \\ J_0 &= +\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_1}-p_1\frac{\partial}{\partial x_1}\right)+\left(x_2\frac{\partial}{\partial p_2}-p_2\frac{\partial}{\partial x_2}\right)\right\}. \end{aligned} \tag{2.13}$$

As in the case of Eq. (2.4), there are six squeeze generators:

$$\begin{aligned} K_1 &= +\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial x_1}\right)+\left(x_2\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial x_2}\right)\right\}, \\ K_2 &= -\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial x_1}\right)-\left(x_2\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial x_2}\right)\right\}, \\ K_3 &= -\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial x_1}-p_1\frac{\partial}{\partial p_1}\right)+\left(x_2\frac{\partial}{\partial x_2}-p_2\frac{\partial}{\partial p_2}\right)\right\}, \\ Q_1 &= -\left(\frac{i}{2}\right)\left\{\left(x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}\right)-\left(p_2\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial p_2}\right)\right\}, \\ Q_2 &= +\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial x_1}-p_1\frac{\partial}{\partial p_1}\right)-\left(x_2\frac{\partial}{\partial x_2}-p_2\frac{\partial}{\partial p_2}\right)\right\}, \\ Q_3 &= -\left(\frac{i}{2}\right)\left\{\left(x_1\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial x_1}\right)+\left(x_2\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial x_2}\right)\right\}. \end{aligned} \tag{2.14}$$

The generators Eq. (2.13) and Eq. (2.14) of course satisfy the commutation relations of Eq. (2.5).

The group Sp(4) has ten parameters, but there are seldom physical problems dealing with ten variables. Thus subgroups of Sp(4) are playing important roles in physics.

III. THREE-PARAMETER SUBGROUPS OF Sp(4)

The group Sp(4) has many subgroups, and three-parameter subgroups are particularly useful in studying two-mode squeezed states.[5,7] The most obvious subgroup is the SU(2)-like group generated by J_1, J_2, and J_3. Another subgroup frequently discussed in the literature is the SU(1,1)-like group generated by J_0, K_3, and Q_3.

From the local isomorphism between Sp(4) and O(3,2), it is not difficult to find all possible SU(1,1)-like subgroups.

Indeed, the above-mentioned SU(1,1)-like subgroup is unitarily equivalent to those generated by J_0, K_1, Q_1 and by J_0, K_2, Q_2. There is also an SU(1,1)-like subgroup generated by K_1, K_2, J_3, which is unitarily equivalent to Q_1, Q_2, and J_3. These are then unitarily equivalent to the subgroups generated by K_2, K_3, J_1, by K_3, K_1, J_2, by Q_2, Q_3, J_1, and by Q_3, Q_1, J_2. It is known from the Lorentz group that the signs of K_i and Q_i can be changed.

The purpose of this paper is to discuss additional three-parameter subgroups. It is known that the O(3,1) Lorentz group has a number of E(2)-like subgroups.[3,8,10] Since there are two O(3,1)-like subgroups in the O(3,2) deSitter group, the group Sp(4) should contain a number of E(2)-like subgroups.[8,9]

The best way to approach this problem is to use the lesson we learned from the Lorentz group.[10,11] If L_1, L_2, L_3 and B_1, B_2, B_3 are the generators of rotations and boosts in O(3,1), there is a subgroup generated by L_3, T_1, and T_2, where

$$T_1=B_1-L_2 \quad \text{and} \quad T_2=B_2+L_1. \tag{3.1}$$

They satisfy the commutation relations:

$$[T_1,T_2]=0, \quad [L_3,T_1]=-iT_2, \quad [L_3,T_2]=iT_1. \tag{3.2}$$

This set of commutation relations is the same as that for the Euclidean group in two-dimensional space or E(2).[10] By considering the symmetry of the O(3,1)-like subgroups of Sp(4), we can construct all E(2)-like subgroups of Sp(4).

If we define

$$F_1=K_1-J_2, \quad F_2=K_2+J_1 \tag{3.3}$$

and

$$G_1=Q_1-J_2, \quad G_2=Q_2+J_1, \tag{3.4}$$

then the resulting commutation relations are

$$[F_1,F_2]=0, \quad [J_3,F_1]=-iF_2, \quad [J_3,F_2]=iF_1, \tag{3.5}$$

and similar relations for G_1, G_2, and J_3. The commutation relations remain invariant when K_i and Q_i change their signs. The subgroup generated by G_1, G_2, and J_3 is unitarily equivalent to that generated by F_1, F_2, and J_3. There are four-additional subgroups generated by F_2, F_3, J_1, by F_3, F_1, J_2, by G_2, G_3, J_1, and by G_3, G_1, J_2. They are all unitarily equivalent. Thus we can study them all by studying one.

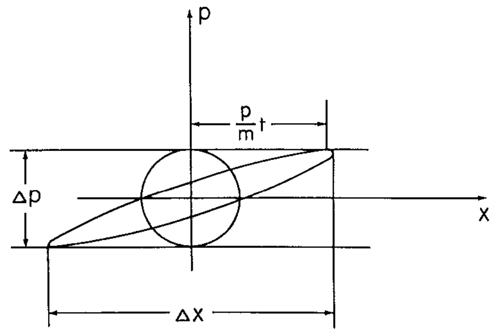

FIG. 1. Illustration of shear. The wave packet spread appears as a shear in phase space in the Wigner phase-space picture of quantum mechanics. The p axis remains invariant, while the x axis expands at the rate of pt/m. Since the p axis remains invariant, this is an xp shear. This transformation leaves the sheared area invariant, and is therefore a canonical transformation. This picture of wave packet spreads is discussed in detail in Ref. 3.

IV. CONSTRUCTION OF SHEARED STATES

Let us define the word "shear." In the xp plane, we can consider linear transformations of the type

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}. \tag{4.1}$$

Under this transformation, the x coordinate undergoes a translation proportional to p while the p variable remains unchanged. This transformation deforms the area in the two-dimensional space of x and p as given in Fig. 1. We shall adopt the convention of calling "xp shear" if x becomes changed while p remains invariant. In this convention, the px shear is represented by

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}. \tag{4.2}$$

For a given function $f(x,p)$, the xp shear leads to $f(x-\alpha p,p)$, while the px shear results in $f(x,p-\beta x)$.

With this preparation, we can go back to the E(2)-like subgroup discussed in Sec. III. The rotation generator J_3 is given in Eq. (2.3), and also in Eq. (2.13). The generators F_1 and F_2 take the matrix form

$$F_1=-i\begin{pmatrix} 0 & \sigma_1 \\ 0 & 0 \end{pmatrix}, \quad F_2=i\begin{pmatrix} 0 & \sigma_3 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

These matrices are nilpotent:

$$(F_1)^2=(F_2)^2=(F_1)(F_2)=0, \tag{4.4}$$

and the transformation matrix becomes

$$S(\alpha,\beta)=\exp(-i\beta F_2)\exp(-i\alpha F_1)$$
$$=I-i\alpha F_1-i\beta F_2, \tag{4.5}$$

which takes the form

$$S(\alpha,\beta)=\begin{pmatrix} I & -\alpha\sigma_1+\beta\sigma_3 \\ 0 & I \end{pmatrix}. \tag{4.6}$$

Let us examine the coordinate transformations due to this matrix. When $\beta=0$, the coordinate transformation becomes

$$x_1'=x_1-\alpha p_2, \quad x_2'=x_2-\alpha p_1. \tag{4.7}$$

This means that $S(\alpha,0)$ performs the x_1p_2 and x_2p_1 shears. Likewise, $S(0,\beta)$ will perform the x_1p_1 and x_2p_2 shears as

$$x_1'=x_1+\beta p_1, \quad x_2'=x_2-\beta p_2. \tag{4.8}$$

The effect of the transformation $S(\alpha,\beta)$ is

$$x_1'=x_1-\alpha p_2+\beta p_1, \quad x_2'=x_2-\alpha p_1-\beta p_2. \tag{4.9}$$

The rotation matrix $R(\phi)=\exp(-i\phi J_3)$ transforms $S(\alpha,0)$ to $S(0,\alpha)$, since

$$\{\exp(-i(\phi/2)\sigma_2)\}\sigma_3\{\exp(i(\phi/2)\sigma_2)\}$$
$$=(\sin\phi)\sigma_3+(\sin\phi)\sigma_1. \tag{4.10}$$

Thus if we take into account rotations generated by J_3, it is sufficient to restrict ourselves to one generator of the form

$$F(\phi)=(\cos\phi)F_2-(\sin\phi)F_1, \tag{4.11}$$

and the transformation matrix can be written as

$$S(\rho,\phi)=\exp(-i\rho F(\phi))=I-i\rho F(\phi), \tag{4.12}$$

where I is the unit matrix, and

$$\alpha=\rho\cos\phi, \quad \beta=\rho\sin\phi. \tag{4.13}$$

As for the rotation operator applied to the coordinate variables, the effect of the transformation is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}=\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix}=\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{4.14}$$

The rotation in this case is the simultaneous parallel rotation of the x_1x_2 and p_1p_2 coordinate systems.

The differential forms of the generators are

$$F_1=i\left(p_2\frac{\partial}{\partial x_1}+p_1\frac{\partial}{\partial x_2}\right),$$
$$F_2=-i\left(p_1\frac{\partial}{\partial x_1}-p_2\frac{\partial}{\partial x_2}\right), \tag{4.15}$$

while J_3 is given in Eq. (2.13). The operation of $S(\alpha,\beta)$ to the function $f(x_1,x_2,p_1,p_2)$ leads to $f(x_1+\alpha p_2-\beta p_1, x_2+\alpha p_1+\beta p_2, p_1, p_2)$. If the above shear operator is applied to the Wigner function of Eq. (2.12),

$$S(\alpha,\beta)W(x_1,x_2,p_1,p_2)$$
$$=(1/\pi)^2\exp\{-(x_1+\alpha p_2-\beta p_1)^2$$
$$-(x_2+\alpha p_1+\beta p_2)^2-(p_1^2+p_2^2)\}. \tag{4.16}$$

This is the Wigner function for the sheared state. As we shall see in Sec. VII, we shall use this function to calculate measurable quantities, including the total number of photons and variances in the photon number in a sheared state of light. Before performing these calculations, we are interested in the distribution of photons in the sheared state.

V. SHEARED STATES IN THE SCHRÖDINGER PICTURE

The generators of Sp(4) given in Sec. II are applicable to the Wigner function of two pairs of canonical variables. Since the Wigner function is constructed from the Schrödinger wave function, it may be possible to construct the generators applicable to the wave function which will lead to the Sp(4) transformations in phase space. Indeed, this question has been addressed,[7] and the corresponding generators are

$$\hat{J}_1=\tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2), \quad \hat{J}_2=\tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1),$$
$$\hat{J}_3=(1/2i)(\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1), \quad \hat{J}_0=\tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2\hat{a}_2^\dagger),$$
$$\hat{K}_1=\tfrac{1}{2}(\hat{a}_1^\dagger\hat{a}_2^\dagger+\hat{a}_1\hat{a}_2),$$
$$\hat{K}_2=-\tfrac{1}{4}(\hat{a}_1^\dagger\hat{a}_1^\dagger+\hat{a}_1\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2^\dagger-\hat{a}_2\hat{a}_2),$$
$$\hat{K}_3=(i/4)(\hat{a}_1^\dagger\hat{a}_1^\dagger-\hat{a}_1\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2^\dagger-\hat{a}_2\hat{a}_2), \tag{5.1}$$
$$\hat{Q}_1=(i/2)(\hat{a}_1^\dagger\hat{a}_2^\dagger-\hat{a}_1\hat{a}_2),$$
$$\hat{Q}_2=-(i/4)(\hat{a}_1^\dagger\hat{a}_1^\dagger-\hat{a}_1\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2^\dagger+\hat{a}_2\hat{a}_2),$$
$$\hat{Q}_3=-(\tfrac{1}{4})(\hat{a}_1^\dagger\hat{a}_1^\dagger+\hat{a}_1\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2^\dagger+\hat{a}_2\hat{a}_2),$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the step-down and step-up operators of the form

$$\hat{a}=(\hat{x}+i\hat{p})/\sqrt{2}, \quad \hat{a}^\dagger=(\hat{x}-i\hat{p})/\sqrt{2}, \tag{5.2}$$

where $\hat{x}$ is the multiplication by x and $\hat{p}$ is the operation of $-i(\partial/\partial x)$. Both operators are applicable to the Schrödinger wave function in the x space. The role of these operators on harmonic oscillator wave functions is well known.

The rotation operator J_3 can be written as

$$\hat{J}_3=-\frac{i}{2}\left\{x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}\right\}. \tag{5.3}$$

Here again, this generates rotations in the x_1x_2 coordinate system. The rotation angle is $\frac{1}{2}$ of that in the case of phase space. The $\hat{F}_1$ and $\hat{F}_2$ operators are

$$\begin{aligned}\hat{F}_1&=\tfrac{1}{2}(\hat{a}_1-\hat{a}_1^\dagger)(\hat{a}_2-\hat{a}_2^\dagger),\\ \hat{F}_2&=\tfrac{1}{4}\{(\hat{a}_2-\hat{a}_2^\dagger)^2-(\hat{a}_1-\hat{a}_1^\dagger)^2\}.\end{aligned} \tag{5.4}$$

Because of the $\frac{1}{2}$ factor in $\hat{J}_3$ in Eq. (5.3), the generator $\hat{F}(\phi)$ of Eq. (4.11) is translated into

$$\begin{aligned}\hat{F}(\phi)&=(\cos\phi)\hat{F}_2+(\sin\phi)\hat{F}_1\\ &=\tfrac{1}{4}(\cos\phi)\{(\hat{a}_2-\hat{a}_2^\dagger)^2-(\hat{a}_1-\hat{a}_1^\dagger)^2\}\\ &\quad+\tfrac{1}{2}(\sin\phi)(\hat{a}_1-\hat{a}_1^\dagger)(\hat{a}_2-\hat{a}_2^\dagger).\end{aligned} \tag{5.5}$$

The shear operator is

$$\hat{S}(\rho,\phi)=\exp(-i\rho\hat{F}(\phi)). \tag{5.6}$$

If $\phi=0$, this operator becomes separable in the x_1 and x_2 coordinate, and the shear operator takes the form

$$\begin{aligned}\hat{S}(\rho,0)&=\exp\{-i(\rho/4)(\hat{a}_2-\hat{a}_2^\dagger)^2\}\\ &\quad\times\exp\{i(\rho/4)(\hat{a}_1-a_1^\dagger)^2\},\end{aligned} \tag{5.7}$$

where the first and second modes are decoupled. If $\phi=90°$, the coupling becomes maximum, and the shear operator becomes

$$\hat{S}(\rho,\pi/2)=\exp\{-i(\rho/2)(\hat{a}_2-\hat{a}_2^\dagger)(\hat{a}_1-\hat{a}_1^\dagger)\}. \tag{5.8}$$

Let us go back to the shear operator of Eq. (5.7). We can construct a sheared state by applying it to the ground state. In the momentum space, the application of this operator results in the multiplication by the exponential factor

$$\exp\{-i(\rho/2)(p_1^2-p_2^2)\}. \tag{5.9}$$

If this factor is multiplied by the wave function

$$\psi_0(p_1,p_2)=(1/\pi)^{1/2}\exp\{-(\tfrac{1}{2})(p_1^2+p_2^2)\} \tag{5.10}$$

corresponding to the ground state, the result is

$$\begin{aligned}\psi_\rho(p_1,p_2)&=\left(\frac{1}{\pi}\right)^{1/2}\exp\left\{-\left(\frac{1+i\rho}{2}\right)(p_1)^2\right\}\\ &\quad\times\exp\left\{-\left(\frac{1-i\rho}{2}\right)(p_2)^2\right\}.\end{aligned} \tag{5.11}$$

As for an arbitrary angle ϕ, the sheared state is

$$\begin{aligned}&\psi_\rho(p_1,p_2)\\ &=\left(\frac{1}{\pi}\right)^{1/2}\exp\left\{-\left(\frac{1+i\rho}{2}\right)\left(p_1\cos\frac{\phi}{2}+p_2\sin\frac{\phi}{2}\right)^2\right\}\\ &\quad\times\exp\left\{-\left(\frac{1-i\rho}{2}\right)\left(p_1\sin\frac{\phi}{2}-p_2\cos\frac{\phi}{2}\right)^2\right\}.\end{aligned} \tag{5.12}$$

If $\phi=90°$, the sheared state becomes

$$\begin{aligned}\psi_\rho(p_1,p_2)&=\left(\frac{1}{\pi}\right)^{1/2}\exp\left\{-\left(\frac{1+i\rho}{4}\right)(p_1+p_2)^2\right\}\\ &\quad\times\exp\left\{-\left(\frac{1-i\rho}{4}\right)(p_1-p_2)^2\right\}.\end{aligned} \tag{5.13}$$

We can obtain the results of Eqs. (5.12) and (5.13) by applying the shear operator of Eq. (5.6) to the ground state. This procedure will be discussed in Sec. VI. The wave functions given in Eqs. (5.11)–(5.13) are not eigenfunctions of the harmonic oscillator Hamiltonian. They can be written as linear expansions of the eigenfunctions. As is well known, the mathematics of harmonic oscillators is applicable to the occupation number space commonly called Fock space. In Fock space, the oscillator energy level becomes the number of particles. The coherent state of photons is a superposition of states with different photon numbers.[1] The oscillator formalism discussed here will lead to another coherent superposition of photon states. In view of the geometric picture discussed in Sec. IV, we shall call them sheared states,

VI. SHEARED STATES IN FOCK SPACE

The mathematics given in Sec. V is directly applicable to occupation number states in Fock space. The nth excited oscillator state corresponds to the state with n free particles, particularly n photons. The step-up and step-down operators are now the creation and annihilation operators. Their role in the process of second quantization is well known. The harmonic oscillator in its ground state corresponds to the vacuum or zero-photon state.

The sheared states given in Sec. V are expandable in terms of number states. The most general form of the shear operator in momentum space can be written as

$$\hat{S}(\alpha,\beta)=\exp\{i(\alpha/2)(p_1^2-p_2^2)+i\beta p_1 p_2\}, \tag{6.1}$$

where α and β are real parameters. We now write the sheared state as

$$|\alpha,\beta\rangle=\hat{S}(\alpha,\beta)\,|0,0\rangle. \tag{6.2}$$

Then

$$\langle p_1,p_2|\alpha,\beta\rangle=\left(\frac{1}{\pi}\right)^{1/2}\exp\left\{-\left(\frac{1-i\alpha}{2}\right)p_1^2 -\left(\frac{1+i\alpha}{2}\right)p_2^2+i\beta p_1 p_2\right\}. \tag{6.3}$$

This wave function is normalized, as the shear $S(\alpha,\beta)$ is a unitary transformation. The remaining task is to expand this expression in terms of the complete orthonormal set of harmonic oscillator wave functions.

If $\beta=0$, the calculation is relatively simple. Indeed,

$$\langle p_1,p_2|\alpha,0\rangle=\left(\frac{1}{\pi}\right)^{1/2}\exp\left\{-\left(\frac{1-i\alpha}{2}\right)p_1^2\right\} \times\exp\left\{-\left(\frac{1+i\alpha}{2}\right)p_2^2\right\}. \tag{6.4}$$

The form

$$g(p)=(1/\pi)^{1/4}\exp\{-((1-i\alpha)/2)p^2\} \tag{6.5}$$

is a wave function with a complex spring constant. It can be expanded as[19]

$$g(p)=(1/\pi)^{1/4}e^{-p^2/2}\sum_{m=0}^{\infty}(2^{2m}(2m)!)^{-1/2}C_{2m}H_n(p), \tag{6.6}$$

where $H_n(p)$ is the Hermite polynomial, and

$$C_{2m}=(-i)^m\left(\frac{\sqrt{(2m)!}}{m!}\right)\left(\frac{\alpha}{4}\right)^m\left(1-i\frac{\alpha}{2}\right)^{-m-1/2} \tag{6.7}$$

Thus the wave function Eq. (6.4) can be expanded as

$$\langle p_1,p_2|\alpha,0\rangle=\left\{\sum_{m=0}^{\infty}C_{2m}\langle p_1|2m\rangle\right\} \times\left\{\sum_{n=0}^{\infty}C_{2n}^*\langle p_2|2n\rangle\right\}. \tag{6.8}$$

If β is not zero, it is possible to express the polynomial in the exponent of Eq. (6.1):

$$\alpha(p_1^2-p_2^2)+2\beta p_1 p_2 \tag{6.9}$$

as

$$\rho\left\{\left(p_1\cos\frac{\phi}{2}+p_2\sin\frac{\phi}{2}\right)^2-\left(p_1\sin\frac{\phi}{2}-p_2\cos\frac{\phi}{2}\right)^2\right\}, \tag{6.10}$$

with $\rho=(\alpha^2+\beta^2)^{1/2}$, $\tan\phi=\beta/\alpha$. As a consequence, the sheared wave function of Eq. (6.3) becomes that of Eq. (6.8), and the wave function is separable in the variables p_1' and p_1',

$$p_1'=p_1\cos\frac{\phi}{2}+p_2\sin\frac{\phi}{2}, \qquad p_2'=-p_1\sin\frac{\phi}{2}+p_2\cos\frac{\phi}{2}, \tag{6.11}$$

as was noted in Eq. (5.12), and the expansion procedure for the $\beta=0$ case can be applied directly to these separable variables. On the other hand, if we choose to expand in the original variables p_1 and p_2, there will be cross terms, and the final result will be a very complicated series. The complication may be beyond our control.

In order to deal with this problem, we take a fresh approach. Let us go back to Eq. (5.5) where the shear operator is written in terms of the creation and annihilation operators. Since the application of the annihilation operator to the vacuum state produces zero, it is natural for us to look for the procedure for writing the shear operator as a normal ordered operator. The shear generator of Eq. (5.5) is quadratic in $\hat{a}_i$ and $\hat{a}_i^\dagger$. It is trivial to write the generator in a normal ordered form, but the problem is entirely different for the shear operator. Indeed, there have been discussions of this problem in the past, and we shall use the result available in the literature.[20]

For the present case, $\rho\hat{F}(\phi)$ can be written as

$$\rho\hat{F}(\phi)=(\alpha/4)((\hat{a}_1^\dagger-\hat{a}_1)^2-(\hat{a}_2^\dagger-\hat{a}_2)^2) +(\beta/2)(\hat{a}_1^\dagger-\hat{a}_1)(\hat{a}_2^\dagger-\hat{a}_2), \tag{6.12}$$

which takes the quadratic form of

$$\rho\hat{F}(\phi)=(\alpha/4)(\hat{a}_1^{\dagger 2}+\hat{a}_1^{2}-2\hat{a}_1^{\dagger}\hat{a}_1) - (\alpha/4)(\hat{a}_2^{\dagger 2}+\hat{a}_2^{2}-2\hat{a}_2^{\dagger}\hat{a}_2)+(\beta/2) \times(\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}+\hat{a}_1\hat{a}_2-\hat{a}_2^{\dagger}\hat{a}_1-\hat{a}_1^{\dagger}\hat{a}_2). \quad (6.13)$$

This is normal ordered, but the shear operator $S(\alpha,\beta) = \exp(-i\rho\hat{F}(\phi))$ is not. However, there are theorems in the literature which allow us to write the $S(\alpha,\beta)$ operator in a normal-ordered form.[20] Indeed,

$$\hat{S}(\alpha,\beta)=\lambda{:}\exp\{\xi(\hat{a}_1^{\dagger 2}+\hat{a}_1^{2}-2\hat{a}_1^{\dagger}\hat{a}_1) +\eta(\hat{a}_2^{\dagger 2}+\hat{a}_2^{2}-2\hat{a}_2^{\dagger}\hat{a}_2) +\zeta(\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}+\hat{a}_1\hat{a}_2-\hat{a}_1^{\dagger}\hat{a}_2-\hat{a}_2^{\dagger}\hat{a}_1)\}{:}, \quad (6.14)$$

where

$$\lambda=2/(\alpha^2+\beta^2+4)^{1/2},$$

$$\xi=(\alpha^2+\beta^2-2i\alpha)/2(\alpha^2+\beta^2+4),$$

$$\eta=(\alpha^2+\beta^2+2i\alpha)/2(\alpha^2+\beta^2+4),$$

$$\zeta=-2i\beta/(\alpha^2+\beta^2+4).$$

If this operator is applied to the vacuum state, the annihilation operators are deleted. The result is

$$\hat{S}(\alpha,\beta)|0,0\rangle=\hat{T}(\alpha,\beta)|0,0\rangle, \quad (6.15)$$

where

$$\hat{T}(\alpha,\beta)=\lambda\exp(\xi\hat{a}_1^{\dagger 2}+\eta\hat{a}_2^{\dagger 2}+\zeta\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}).$$

This operator can now be decomposed into

$$\hat{T}(\alpha,\beta)=\lambda\exp(\xi\hat{a}_1^{\dagger 2})\exp(\eta\hat{a}_2^{\dagger 2})\exp(\zeta\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}). \quad (6.16)$$

If this operator is acted on the vacuum state,

$$|\alpha,\beta\rangle=\hat{T}(\alpha,\beta)|0\rangle=\sum_{k,j} C_{kj}|k,j\rangle, \quad (6.17)$$

where k and j are the Fock-space indices for the p_1 and p_2 coordinates, respectively. For the system of photons, they are the photon numbers for the first and second kinds, respectively. Using the form of Eq. (6.16), it is possible to calculate the coefficient C_{kj}. The results are

$$C_{2m+1,2n}=C_{2m,2n+1}=0,$$

$$C_{2m,2n} =\lambda\sqrt{(2m)!(2n)!}\sum_{k=0}^{\mathrm{Min}(m,n)}\frac{\zeta^{2k}\xi^{m-k}\eta^{n-k}}{(2k)!(m-k)!(n-k)!},$$

$$C_{2m+1,2n+1}=\lambda\sqrt{(2m+1)!(2n+1)!} \times\sum_{k=0}^{\mathrm{Min}(m,n)}\frac{\zeta^{2k+1}\xi^{m-k}\eta^{n-k}}{(2k+1)!(m-k)!(n-k)!}. \quad (6.18)$$

When $\beta=0$, $C_{2m+1,2n+1}=0$, and the series of Eq. (6.17) becomes the separable expansion of Eq. (6.8).

Indeed, these coefficients are needed for evaluating the distribution in photons numbers. If needed, we can plot a three-dimensional plot of the photon distribution in two variables using a computer program such as MATHEMATICA. The distribution $|C_{k,j}|^2$ of course depends on k and j, which are the numbers for the first and second photons. These coefficients are normalized. The average value of the number of the first and second photons are

$$\langle N_1\rangle=\sum_{k=0}^{\infty} k|C_{k,j}|^2, \quad \langle N_2\rangle=\sum_{j=0}^{\infty} j|C_{k,j}|^2. \quad (6.19)$$

Similarly,

$$\langle N_1N_2\rangle=\sum_{k=0}^{\infty} kj|C_{k,j}|^2, \quad (6.20)$$

and

$$\langle N_1^2\rangle=\sum_{k=0}^{\infty} k^2|C_{k,j}|^2, \quad \langle N_2^2\rangle=\sum_{j=0}^{\infty} j^2|C_{k,j}|^2. \quad (6.21)$$

The computation of these numbers in terms of the coefficients given in Eq. (6.18) is possible. On the other hand, it is not clear whether this computation will lead to a closed analytical form for each of the above quantities.

It is of course possible to obtain analytical expressions using the the momentum wave function given in Eq. (6.3), and the expression of the operators in terms of the momentum variables. It is also possible to compute these numbers using the sheared Wigner function. While the computation of expectation values in the Schrödinger picture is a well-known procedure, the method based in the phase-space picture is relatively new.[3] We shall use this new method to compute the measurable quantities in Sec. VII.

VII. PHOTON NUMBER AND ITS VARIANCE

The method of calculating measurable numbers from the Wigner function has been discussed in the litera-ture.[3,18,21] In particular, the photon number can be calculated from the expression

$$\langle N\rangle=\frac{1}{2}\int (x^2+p^2-1)W(x,p)dx\,dp, \tag{7.1}$$

and the $(\text{number})^2$ from

$$\langle N^2\rangle=\frac{1}{4}\int \{(x^2+p^2-1)^2-1\}W(x,p)dx\,dp \tag{7.2}$$

for the single-mode state.

It is straightforward to generalize the above procedure to two-mode states. The total number of photon is $N = N_1 + N_2$. In terms of the Wigner function,

$$\langle N\rangle=\frac{1}{2}\int (x_1^2+p_1^2+x_2^2+p_2^2-2)W(x_1,x_2,p_1,p_2)\times dx_1\,dp_1\,dx_2\,dp_2. \tag{7.3}$$

Since

$$N^2=(N_1+N_2)^2=N_1^2+N_2^2+2N_1N_2, \tag{7.4}$$

the application of Eqs. (7.1) and (7.2) to each mode leads to

$$\langle N^2\rangle=\int \left\{\frac{1}{4}(x_1^2+p_1^2+x_2^2+p_2^2)^2-(x_1^2+p_1^2+x_2^2+p_2^2)+\frac{1}{2}\right\} W(x_1,x_2,p_1,p_2)dx_1\,dp_1\,dx_2\,dp_2. \tag{7.5}$$

It is now straightforward to calculate $\langle N\rangle$ and $\langle N^2\rangle$ using the Wigner function of Eq. (4.16). The results are

$$\langle N\rangle=\tfrac{1}{2}(\alpha^2+\beta^2),$$
$$\langle N^2\rangle=\tfrac{1}{2}\{(\alpha^2+\beta^2)^2+2(\alpha^2+\beta^2)+2\alpha^2\beta^2\}. \tag{7.6}$$

From this, we can calculate the variance:

$$\langle(\Delta N)^2\rangle=\langle N^2\rangle-\langle N\rangle^2=\tfrac{1}{4}\{(\alpha^2+\beta^2)^2+4(\alpha^2+\beta^2)+4\alpha^2\beta^2\}. \tag{7.7}$$

Naturally, if $\alpha=\beta=0$, there are no shears, and the vacuum expectation values of N, N^2, and the variance are zero. When $(\alpha^2+\beta^2)^2$ becomes very large, the ration $\langle(\Delta N)^2\rangle/\langle N^2\rangle$ becomes 1/2.

VIII. FURTHER COMMENTS ON SHEARED STATES

Among the fundamental transformations in physics, we are most familiar with rotations and unitary transformations. Recently, due to the theoretical development in quantum optics and the phase-space approach in classical mechanics, the squeeze is now being regarded as one of the fundamental transformations. However, since the Lorentz boost is a squeeze transformation,[3] the geometrical concept of squeeze is not really new in physics. On the other hand, it is new that this concept leads to new distributions of photons in quantum optics.

In this paper, we discussed the concept of shear applicable to photon distributions. From the geometrical point of view, the shear transformation is quite common in engineering sciences. It was recently observed that this geometrical operation corresponds to translation-like transformations in Wigner's little group for massless particles whose translation-like transformations correspond to gauge transformations.[9,11] It is interesting to note that photon distributions in the sheared state and the Wigner gauge transformation share the same mathematics.

From the mathematical point of view, the two-mode squeezed state is produced from a product of two ground-state oscillator wave functions with the same frequency ω_0.[3,4,15] Then the frequencies ω_1 and ω_2 are squeezed in such a way that the product $\omega_1\omega_2$ remain invariant. In the case of shear operations, we start also from the two ground-state wave functions with the same frequency. It is interesting to note from Secs. V and VI of this paper that the shear operation adds an imaginary number to one of the frequencies while subtracting the same imaginary number from the other frequency. The net result is that the shear transformation preserves the relation $\omega_1+\omega_2=2\omega_0$, and $\omega_1 = \omega_2^*$. The harmonic oscillator with a complex frequency has been discussed in the literature.[22]

One of the most interesting features of the two-dimensional Euclidean group is that it can be obtained from the contraction of O(3) or O(2,1). They are are locally isomorphic to SU(2) and SU(1,1), respectively.[12,13] Therefore, we should be able to obtain the result of the present paper by applying certain limiting procedure to the SU(2) and SU(1,1) coherent states. The first step toward this ambitious program is to construct the E(2)-symmetric coherent state. This is precisely what we have done in this paper.

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