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Correspondence between the Classical and Quantum Canonical Transformation Groups from an Operator Formulation of the Wigner Function

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An explicit expression of the "Wigner operator" is derived, such that the Wigner function of a quantum state is equal to the expectation value of this operator with respect to the same state. This Wigner operator leads to a representation-independent procedure for establishing the correspondence between the inhomogeneous symplectic group applicable to linear canonical transformations in classical mechanics and the Weyl–metaplectic group governing the symmetry of unitary transformations in quantum mechanics.

1. INTRODUCTION

If the Hamiltonian of a classical system is (inhomogeneously) quadratic in the canonical coordinates and/or momenta, the quantum version of this system always possesses a semiclassical behavior.[1] Therefore, these kinds of Hamiltonians occupy a special place in the study of the correspondence between classical and quantum mechanics. The purpose of this paper is to investigate the correspondence between the linear canonical transformations (L.C.T.'s) in classical mechanics and those in quantum mechanics, by developing the Wigner operator formalism.

Classical mechanics and quantum mechanics are based on different mathematical frameworks. Indeed, there have been many attempts in the past to establish the connections between them. In this paper, we use the Weyl–Wigner–Moyal formalism[2] to study this problem. The advantage of

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this formalism is that it associates a quantum state in Hilbert space with a real-valued function, namely the Wigner (quasi-probability distribution) function[3-5] which permits a phase-space picture of quantum mechanics.[6-9]

There are a number of equivalent definitions for canonical transformations in classical mechanics.[10-12] Among them, the Poisson-bracket formalism possesses an explicit quantum analogue. Since the quantum counterparts of Poisson brackets of canonical variables are the commutators of the corresponding operators, it appears to us that the quantum L.C.T. is the transformation of operators which preserves the canonical commutation relations, e.g., the Bogoliubov transformation.[4,13,14] If we adopt this point of view, there is a one-to-one correspondence between the classical L.C.T.'s and quantum L.C.T.'s, because the only change from the classical to the quantum case is the replacement of the canonical variables by those corresponding operators.

Since, in quantum mechanics, the physics is represented by states as well as operators, the correct definition of the quantum L.C.T. is an operation which produces both an L.C.T. of operators and a unitary transformation of states. It will be shown that this kind of operations can be performed with the unitary operators constructed from the position and momentum operators. The group of these unitary operators is the quantum canonical transformation group.

For a physical system with n degrees of freedom, the symmetry group for classical L.C.T.'s is the inhomogeneous symplectic group $\mathbf{ISp(2n, R)}$. It is the semidirect product of the translation group $\mathbf{T(2n)}$ and the symplectic group $\mathbf{Sp(2n, R)}$. The quantum analogue of $\mathbf{T(2n)}$ in phase space is the Weyl (or Weyl–Heisenberg) group $\mathbf{W(2n)}$, and the symplectic group has the metaplectic group $\mathbf{Mp(2n, R)}$ as its quantum counterpart.[1,15,16] The semidirect product of $\mathbf{W(2n)}$ and $\mathbf{Mp(2n, R)}$, which will be denoted by $\mathbf{WMp(2n, R)}$, is the quantum canonical transformation group. The important point is that the structure of $\mathbf{WMp(2n, R)}$ is different from that of $\mathbf{ISp(2n, R)}$.

In this paper, we study the correspondence between these two groups by deriving an explicit operator formalism of the phase-space picture of quantum mechanics, in which the Wigner function is the expectation value of the "Wigner operator." The form of this Wigner operator is surprisingly simple but has rich mathematical implications. By using this Wigner operator, we establish the correspondence between $\mathbf{ISp(2n, R)}$ and $\mathbf{WMp(2n, R)}$. Since the Wigner operator is independent of representations and states, the correspondence is also independent of them.

In Sec. 2, notations, conventions, and definitions are introduced for the mathematical symbols and words used in this paper. In Sec. 3, we define and compute the Wigner operator whose expectation value gives the

Wigner function. In Secs. 4 and 5, the inhomogeneous symplectic group and the Weyl–metaplectic group are discussed as the symmetry groups in classical and quantum mechanics, respectively. In Secs. 6 and 7, we discuss the correspondence between these two groups in detail.

2. NOTATIONS, CONVENTIONS, AND DEFINITIONS

Throughout this paper, $\hbar$ is set equal to 1; the Einstein summation convention is followed with indices running from 1 to n; "$*$" denotes complex conjugate, "$\dagger$" denotes Hermitian conjugate, and "T" denotes the transpose of a matrix.

We use $\mathbf{x}=(x_1, x_2,..., x_n)$ and $\mathbf{k}=(k_1, k_2,..., k_n)$ for the n-dimensional canonical coordinate and momentum, respectively. Thus $(\mathbf{x}, \mathbf{k})$ is a vector in the $2n$-dimensional phase space. $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ denote the n-dimensional position and momentum operators corresponding to the canonical variables $\mathbf{x}$ and $\mathbf{k}$. The canonical commutation relations (CCR's) are

$$[\hat{q}_i, \hat{q}_j]=[\hat{p}_i, \hat{p}_j]=0, \qquad [\hat{q}_i, \hat{p}_j]=i\delta_{ij} \tag{1}$$

The physical system under consideration is of n degrees of freedom with its time-independent Hamiltonian (inhomogeneously) quadratic in $(\mathbf{x}, \mathbf{k})$ or $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$. $\hat{\mathbf{a}}=(1/\sqrt{2})(\hat{\mathbf{q}}+i\hat{\mathbf{p}})$ and $\hat{\mathbf{a}}^\dagger=(1/\sqrt{2})(\hat{\mathbf{q}}-i\hat{\mathbf{p}})$ are the annihilation and creation operators, while $\boldsymbol{\alpha}=(1/\sqrt{2})(\mathbf{x}+i\mathbf{k})$ and $\boldsymbol{\alpha}^*=(1/\sqrt{2})(\mathbf{x}-i\mathbf{k})$ are the classical counterparts of $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$. $|0\rangle$ denotes the n-mode Fock vacuum state, i.e., the ground state of an n-dimensional harmonic oscillator with unit mass and frequencies,

$$\langle \mathbf{x} | 0\rangle=\pi^{-n/4}\exp(-\tfrac{1}{2}\mathbf{x}^2) \tag{2}$$

The (phase space) displacement operator $\hat{D}[(\mathbf{x}, \mathbf{k})]=\hat{D}(\boldsymbol{\alpha})$ is defined as

$$\begin{aligned}\hat{D}[(\mathbf{x}, \mathbf{k})]&=\exp[i(\mathbf{k}\cdot\hat{\mathbf{q}}-\mathbf{x}\cdot\hat{\mathbf{p}})]=\exp[i(\hat{\mathbf{q}}, \hat{\mathbf{p}})\,\mathscr{J}(\mathbf{x}, \mathbf{k})^{\mathrm{T}}]\\ &=\hat{D}(\boldsymbol{\alpha})=\exp[\boldsymbol{\alpha}\cdot\hat{\mathbf{a}}^\dagger-\boldsymbol{\alpha}^*\cdot\hat{\mathbf{a}}]\end{aligned} \tag{3}$$

where

$$\mathscr{J}=\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ -\mathbf{1} & \mathbf{0}\end{pmatrix}, \qquad \mathbf{1}=n\times n \text{ unit matrix} \tag{4}$$

The Klein operator (or parity operator) is defined as $\exp(i\pi\hat{N})=(-1)^{\hat{N}}$, where

$$\hat{N}=\sum_{k=1}^{n}\tfrac{1}{2}(\hat{p}_k^2+\hat{q}_k^2-1)=\sum_{k=1}^{n}\hat{a}_k^\dagger\hat{a}_k \tag{5}$$

Both $\hat{D}[(\mathbf{x}, \mathbf{k})]$ and $\exp(i\pi\hat{N})$ are unitary and have the following properties:

$$\hat{D}^{\dagger}[(\mathbf{x}, \mathbf{k})] = \hat{D}^{-1}[(\mathbf{x}, \mathbf{k})] = \hat{D}[-(\mathbf{x}, \mathbf{k})] \tag{6}$$

$$\exp(i\pi\hat{N})^{\dagger} = \exp(i\pi\hat{N})^{-1} = \exp(-i\pi\hat{N}) = \exp(i\pi\hat{N}) \tag{7}$$

The coherent state is defined as[17]

$$|(\mathbf{x}, \mathbf{k})\rangle = \hat{D}[(\mathbf{x}, \mathbf{k})]\,|0\rangle = |\boldsymbol{\alpha}\rangle = \hat{D}(\boldsymbol{\alpha})\,|0\rangle \tag{8}$$

hence

$$\langle(\mathbf{x}, \mathbf{k})| = \langle\boldsymbol{\alpha}| = \langle 0|\,\hat{D}[-(\mathbf{x}, \mathbf{k})] \tag{9}$$

From the above definitions, we can derive the following relations which will be useful in later discussions:

$$\exp(i\pi\hat{N})\,|0\rangle = |0\rangle \tag{10}$$

$$\langle 0|\,\hat{D}[(\mathbf{x}, \mathbf{k})]\,|0\rangle = \exp[-\tfrac{1}{4}(\mathbf{x}^2 + \mathbf{k}^2)] \tag{11}$$

$$\hat{D}[(\mathbf{x}_1, \mathbf{k}_1)]\,\hat{D}[(\mathbf{x}_2, \mathbf{k}_2)]\,\hat{D}[(\mathbf{x}_1, \mathbf{k}_1)] = \hat{D}[(2\mathbf{x}_1 + \mathbf{x}_2, 2\mathbf{k}_1 + \mathbf{k}_2)] \tag{12}$$

$$\hat{D}[(\mathbf{x}, \mathbf{k})](\hat{\mathbf{q}}, \hat{\mathbf{p}})\,\hat{D}[-(\mathbf{x}, \mathbf{k})] = (\hat{\mathbf{q}} - \mathbf{x}, \hat{\mathbf{p}} - \mathbf{k}) \tag{13}$$

$$\exp(i\pi\hat{N})(\hat{\mathbf{q}}, \hat{\mathbf{p}})\exp(-i\pi\hat{N}) = -(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \tag{14}$$

$$\exp(i\pi\hat{N})\,\hat{D}[(\mathbf{x}, \mathbf{k})]\exp(-i\pi\hat{N}) = \hat{D}[-(\mathbf{x}, \mathbf{k})] \tag{15}$$

3. OPERATOR FORMULATION OF THE WIGNER FUNCTION

If $|\psi\rangle$ is a state of a quantum system with n degrees of freedom, then the Wigner function of this state is defined as[3–5]

$$W_{\psi}(\mathbf{x}, \mathbf{k}) = \pi^{-n} \int_{-\infty}^{\infty} d^n y \exp(2i\mathbf{k}\cdot\mathbf{y})\,\psi^*(\mathbf{x} + \mathbf{y})\,\psi(\mathbf{x} - \mathbf{y}) \tag{16}$$

where $\psi(\mathbf{x}) = \langle\mathbf{x}\,|\,\psi\rangle$ is the wave function in the coordinate representation.

It is clear that we can rewrite the Wigner function in the following representation-independent form:

$$W_{\psi}(\mathbf{x}, \mathbf{k}) = \langle\psi|\,\hat{\Delta}_W[(\mathbf{x}, \mathbf{k})]\,|\psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\psi|\,\Delta_W[(\mathbf{x}, \mathbf{k})]) \tag{17}$$

where

$$\hat{\Delta}_W[(\mathbf{x}, \mathbf{k})] = \pi^{-n} \int_{-\infty}^{\infty} d^n y \exp(2i\mathbf{k}\cdot\mathbf{y})\,|\mathbf{x} + \mathbf{y}\rangle\langle\mathbf{x} - \mathbf{y}| \tag{18}$$

is a well-defined Hermitian operator with $(\mathbf{x}, \mathbf{k})$ as its parameters.

Theorem.

$$\hat{\Delta}_W[(\mathbf{x}, \mathbf{k})] = \pi^{-n}\hat{D}[2(\mathbf{x}, \mathbf{k})] \exp(i\pi\hat{N}) \tag{19}$$

Proof. We give here the proof for $n = 1$. The generalization to arbitrary n is trivial. Define a set of coherent states $|(x_0, k_0)\rangle = \hat{D}[(x_0, k_0)]\,|0\rangle$, where x_0 and k_0 are arbitrary real numbers. The Wigner function of $|(x_0, k_0)\rangle$ can be calculated directly from the definition[9,18]:

$$W(x, k) = \pi^{-1} \exp[-(x - x_0)^2 - (k - k_0)^2] \tag{20}$$

Equating (17) and (20), this yields

$$\begin{aligned}
&\langle (x_0, k_0)|\, \hat{\Delta}_W[(x, k)]\, |(x_0, k_0)\rangle \\
&= \pi^{-1} \exp[-(x - x_0)^2 - (k - k_0)^2] \\
&= \pi^{-1}\langle 0|\, \hat{D}[2(x - x_0, k - k_0)]\, |0\rangle \\
&= \pi^{-1}\langle 0|\, \hat{D}[-(x_0, k_0)]\, \hat{D}[2(x, k)]\, \hat{D}[-(x_0, k_0)]\, |0\rangle \\
&= \pi^{-1}\langle 0|\, \hat{D}[-(x_0, k_0)]\, \hat{D}[2(x, k)] \exp(i\pi\hat{N})\, \hat{D}[(x_0, k_0)] \exp(-i\pi N)\, |0\rangle \\
&= \pi^{-1}\langle (x_0, k_0)|\, \hat{D}[2(x, k)] \exp(i\pi\hat{N})\, |(x_0, k_0)\rangle
\end{aligned} \tag{21}$$

Therefore

$$\langle (x_0, k_0)|\, (\hat{\Delta}_W[(x, k)] - \pi^{-1}\hat{D}[2(x, k)] \exp(i\pi\hat{N}))\, |(x_0, k_0)\rangle = 0 \tag{22}$$

From the overcompleteness of the coherent state basis $\{|(x_0, k_0)\rangle \mid x_0, k_0 \in R\}$, we can conclude that[17]

$$\hat{\Delta}_W[(x, k)] - \pi^{-1}\hat{D}[2(x, k)] \exp(i\pi\hat{N}) \tag{23}$$

must be identically equal to zero, and hence the theorem is proved. Equivalent results have been derived in the literature by different methods.[19-23]

The above result can be directly generalized to the Wigner function of a general (pure or mixed) density operator $\hat{\rho}$ using (17):

$$W_{\hat{\rho}}(\mathbf{x}, \mathbf{k}) = \mathrm{Tr}(\hat{\rho}\Delta_W[(\mathbf{x}, \mathbf{k})]) \tag{24}$$

4. INHOMOGENEOUS SYMPLECTIC GROUP

In this section, we define the group which corresponds to the L.C.T. in the $2n$-dimensional phase space. We adopt the definition of canonical transformations as the transformations which preserve the Poisson brackets of $(\mathbf{x}, \mathbf{k})$. According to this definition, the L.C.T. contains the following two kinds of transformations:

(1) Translation in phase space:

$$(\mathbf{x}, \mathbf{k})^{\mathrm{T}} \to (\mathbf{x} - \mathbf{x}_0, \mathbf{k} - \mathbf{k}_0)^{\mathrm{T}} \tag{25}$$

The group which corresponds to this transformation is the translation group **T(2n)**. It is a $2n$-dimensional Abelian Lie group.

(2) Symplectic transformation:

$$(\mathbf{x}, \mathbf{k})^{\mathrm{T}} \to \mathscr{M}(\mathbf{x}, \mathbf{k})^{\mathrm{T}} \tag{26}$$

where $\mathscr{M}$ is a $2n \times 2n$ real matrix and $\mathscr{M}^{\mathrm{T}} \mathscr{J} \mathscr{M} = \mathscr{J}$, i.e., $\mathscr{M}$ is an element of the matrix group **Sp(2n, R)**.[24] The group which corresponds to this transformation is the symplectic group and is also denoted by **Sp(2n, R)**. It is an $n(2n+1)$-dimensional Lie group.[10]

It is obvious that the combination of the above two transformations gives the most general L.C.T. in the $2n$-dimensional phase space, and the corresponding group is the semidirect product of **T(2n)** and **Sp(2n, R)**, $\mathbf{T(2n)} \otimes_s \mathbf{Sp(2n, R)}$. We will call this group the inhomogeneous symplectic group and denote it by **ISp(2n, R)**. The action of **ISp(2n, R)** on $(\mathbf{x}, \mathbf{k})$ is defined as

$$(\mathbf{x}, \mathbf{k})^{\mathrm{T}} \to \mathscr{M}(\mathbf{x} - \mathbf{x}_0, \mathbf{k} - \mathbf{k}_0)^{\mathrm{T}} \tag{27}$$

5. WEYL–METAPLECTIC GROUP

According to the discussion in Sec. 1, we define the quantum analogue of **ISp(2n, R)** as the group of unitary operators which produce both L.C.T.'s of the operators $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ and unitary transformations of quantum states. From group theory and CCR's, we know that the elements of the Lie algebra of this group must be linear or quadratic in $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$. However, since $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$, the vector space spanned by $\{\hat{q}_i, \hat{p}_j, \hat{q}_i\hat{q}_j, \hat{p}_i\hat{p}_j, \hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i\}$ does not close a Lie algebra. We must introduce a new basis $\hat{I}$ (the identity operator) and redefine $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}\hat{I}$; then the space spanned by $\{\hat{I}, \hat{q}_i, \hat{p}_j, \hat{q}_i\hat{q}_j, \hat{p}_i\hat{p}_j, \hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i\}$ closes a Lie algebra by CCR's. It will be shown below that the Lie group corresponding to this algebra is the quantum analogue of **ISp(2n, R)**.

First, we shall study the quantum analogue of **T(2n)**. It is a $(2n+1)$-dimensional Lie group with its algebra spanned by $\{\hat{I}, \hat{q}_i, \hat{p}_j\}$, i.e., the Weyl (or Weyl–Heisenberg) algebra. We will call this group the Weyl group and denote it by **W(2n)**. It is a central extension of the Abelian group **T(2n)**.[25]

The elements of **W(2n)** are the unitary operators with the form

$$\begin{aligned}\hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0) &= \exp[i(\theta\hat{I} + \mathbf{k}_0 \cdot \hat{\mathbf{q}} - \mathbf{x}_0 \cdot \hat{\mathbf{p}})] \\ &= \exp(i\theta\hat{I}) \exp[i(\mathbf{k}_0 \cdot \hat{\mathbf{q}} - \mathbf{x}_0 \cdot \hat{\mathbf{p}})] \\ &= \exp(i\theta\hat{I})\, \hat{D}[(\mathbf{x}_0, \mathbf{k}_0)] \qquad (28)\end{aligned}$$

where θ is a real number. The action of $\hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)$ on $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ is defined as

$$\hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathsf{T}}\, \hat{T}^{-1}(\theta, \mathbf{x}_0, \mathbf{k}_0) = (\hat{\mathbf{q}} - \mathbf{x}_0, \hat{\mathbf{p}} - \mathbf{k}_0)^{\mathsf{T}} \qquad (29)$$

This operation is formally isomorphic to (25), the translation in phase space made by the group **T(2n)**; therefore we obtain the following group isomorphism:

$$\mathbf{W(2n)}/\{\exp(i\theta\hat{I})\} \cong \mathbf{W(2n)/U(1)} \cong \mathbf{T(2n)} \qquad (30)$$

Next, we study the quantum analogue of **Sp(2n, R)**. It is an $n(2n+1)$-dimensional Lie group with its algebra spanned by $\{\hat{q}_i\hat{q}_j, \hat{p}_i\hat{p}_j, \hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i\}$. We shall show that this Lie algebra is isomorphic to **sp(2n, r)**—the Lie algebra of **Sp(2n, R)**—and thus this group acquired the name metaplectic group **Mp(2n, R)**.[1,15,16] The elements of the Lie algebra of **Mp(2n, R)** are the anti-Hermitian operators with the form

$$\begin{aligned}\hat{\Phi}(m) &= \frac{i}{2}\,[\alpha_{ij}\hat{q}_i\hat{q}_j + \beta_{ij}\hat{p}_i\hat{p}_j + \gamma_{ij}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)] \\ &= \frac{i}{2}\,(\hat{\mathbf{q}}, \hat{\mathbf{p}}) \begin{pmatrix} \alpha & \gamma \\ \gamma^{\mathsf{T}} & \beta \end{pmatrix} (\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathsf{T}} \\ &= \frac{i}{2}\,(\hat{\mathbf{q}}, \hat{\mathbf{p}})\, \mathscr{J} m (\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathsf{T}} \qquad (31)\end{aligned}$$

where $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \beta_{ji}$, and

$$m = \begin{pmatrix} -\gamma^{\mathsf{T}} & -\beta \\ \alpha & \gamma \end{pmatrix} \in \mathbf{sp(2n, r)} \qquad (32)$$

is a $2n \times 2n$ real matrix.[24]

From CCR's, we have

$$\begin{aligned}[\hat{\Phi}(m), (\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathsf{T}}] &= \begin{pmatrix} \gamma^{\mathsf{T}} & \beta \\ -\alpha & -\gamma \end{pmatrix} (\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathsf{T}} \\ &= -m(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathsf{T}} \qquad (33)\end{aligned}$$

and

$$[\hat{\Phi}(m_1), \hat{\Phi}(m_2)] = \hat{\Phi}([m_1, m_2]) \tag{34}$$

Therefore the Lie algebra of **Mp(2n, R)** is isomorphic to **sp(2n, r)**.

The action of $\exp[\hat{\Phi}(m)] \in$ **Mp(2n, R)** on $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ can be defined and calculated from (33):

$$\exp[\hat{\Phi}(m)](\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathrm{T}} \exp[-\hat{\Phi}(m)] = \exp(-m)(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathrm{T}} \tag{35}$$

where $\exp(-m) \in$ **Sp(2n, R)**. Hence this action induces an element in **Sp(2n, R)**.

Let us next generalize (35). We replace the $\exp(-m)$ in (35) by a general element $\mathscr{M}$ in **Sp(2n, R)** and then try to find a unitary operator $\hat{U}(\mathscr{M})$ in **Mp(2n, R)** such that

$$\hat{U}(\mathscr{M})(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathrm{T}} \hat{U}^{-1}(\mathscr{M}) = \mathscr{M}(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\mathrm{T}} \tag{36}$$

From linear algebra and group theory, we know that there is a unique polar decomposition $\mathscr{M} = \mathscr{R}\mathscr{S}$ for any element $\mathscr{M}$ in **Sp(2n, R)**, where $\mathscr{R}$ is orthogonal, $\mathscr{S}$ is symmetric and positive definite, and both $\mathscr{R}$ and $\mathscr{S}$ are in **Sp(2n, R)**. Therefore we can always put $\mathscr{M} = \exp(m_{\mathscr{R}}) \exp(m_{\mathscr{S}})$, where $\mathscr{R} = \exp(m_{\mathscr{R}})$, $\mathscr{S} = \exp(m_{\mathscr{S}})$, and both $m_{\mathscr{R}}$ and $m_{\mathscr{S}}$ are elements in **sp(2n, r)** ($m_{\mathscr{S}}$ is symmetric and unique, while $m_{\mathscr{R}}$ is antisymmetric and not unique).(1) The element $\hat{U}(\mathscr{M})$ in **Mp(2n, R)**, which is unitary and satisfies (36), can be constructed as

$$\hat{U}(\mathscr{M}) = \exp[\hat{\Phi}(-m_S)] \exp[\hat{\Phi}(-m_R)] \tag{37}$$

However, among all the elements in **Mp(2n, R)**, there are exactly two which give the same matrix $\mathscr{M}$ in (36), i.e., $\pm\hat{U}(\mathscr{M})$. The reason that $-\hat{U}(\mathscr{M})$ also belongs to **Mp(2n, R)** is because of the identity

$$\exp[i\pi(\hat{p}_k^2 + \hat{q}_k^2)] = -\hat{I} \in \mathbf{Mp(2n, R)}, \qquad k = 1, 2, \ldots, n \tag{38}$$

Therefore we see that **Mp(2n, R)** is a doubly covering group of **Sp(2n, R)**,

$$\mathbf{Mp(2n, R)}/\{\pm\hat{I}\} \cong \mathbf{Sp(2n, R)} \tag{39}$$

Now we are ready to study the group which corresponds to **ISp(2n, R)**. It is the semidirect product of **W(2n)** and **Mp(2n, R)**, $\mathbf{W(2n)} \otimes_s \mathbf{Mp(2n, R)}$. We will denote this group by **WMp(2n, R)** and define its element as the product

$$\hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \hat{U}(\mathscr{M}) \tag{40}$$

The transformation of $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ under **WMp(2n, R)** is the L.C.T. defined as

$$\begin{aligned}\hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \hat{U}(\mathcal{M})(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\top} [\hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \hat{U}(\mathcal{M})]^{-1} \\ = \hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \hat{U}(\mathcal{M})(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\top}\, \hat{U}^{-1}(\mathcal{M})\, \hat{T}^{-1}(\theta, \mathbf{x}_0, \mathbf{k}_0) \\ = \hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \mathcal{M}(\hat{\mathbf{q}}, \hat{\mathbf{p}})^{\top}\, \hat{T}^{-1}(\theta, \mathbf{x}_0, \mathbf{k}_0) \\ = \mathcal{M}(\hat{\mathbf{q}} - \mathbf{x}_0, \hat{\mathbf{p}} - \mathbf{k}_0)^{\top} \end{aligned} \tag{41}$$

This operation is formally isomorphic to (27), the classical L.C.T. of $(\mathbf{x}, \mathbf{k})$ in phase space.

6. CORRESPONDENCE BETWEEN ISp(2n, R) AND WMp(2n, R) THROUGH THE WIGNER FUNCTION

In Sec. 5, we have shown that the transformation of the operators $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ under **WMp(2n, R)** is the L.C.T. of these operators, which is *formally* isomorphic to the classical L.C.T. in phase space. Hence the action of **WMp(2n, R)** on $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ is exactly the same as **ISp(2n, R)** on the phase space vector $(\mathbf{x}, \mathbf{k})$. Nevertheless, as we have mentioned in Sec. 1, there are two parts to the quantum analogue of a classical L.C.T., i.e., the L.C.T. of operators and the unitary transformation of states. In fact, it is the latter which will produce a machinery enabling us to make a *direct* connection between **ISp(2n, R)** and **WMp(2n, R)**. Since the classical analogue of the quantum state is the Wigner function, the natural way to establish this machinery is via the Wigner function/operator, i.e., to find out the corresponding action of **ISp(2n, R)** on the Wigner function/operator which corresponds to the action of **WMp(2n, R)** on the quantum state.

Let us start with an arbitrary quantum state $|\psi\rangle$. The Wigner function of this state is

$$W_{\psi}(\mathbf{x}, \mathbf{k}) = \langle\psi|\, \hat{\Delta}_W[(\mathbf{x}, \mathbf{k})]\, |\psi\rangle \tag{42}$$

We then transform this state by an element in **WMp(2n, R)** and get

$$|\psi'\rangle = \hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \hat{U}(\mathcal{M})\, |\psi\rangle \tag{43}$$

The transformed Wigner function corresponding to $|\psi'\rangle$ is

$$\begin{aligned}W_{\psi'}(\mathbf{x}, \mathbf{k}) = \langle\psi|\, \hat{U}^{-1}(\mathcal{M})\, \hat{T}^{-1}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \hat{\Delta}_W[(\mathbf{x}, \mathbf{k})] \\ \times \hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\, \hat{U}(\mathcal{M})\, |\psi\rangle \end{aligned} \tag{44}$$

Since

$$\begin{aligned}
&\hat{U}^{-1}(\mathcal{M})\,\hat{T}^{-1}(\theta, \mathbf{x}_0, \mathbf{k}_0)\,\hat{\Delta}_W[(\mathbf{x}, \mathbf{k})]\,\hat{T}(\theta, \mathbf{x}_0, \mathbf{k}_0)\,\hat{U}(\mathcal{M}) \\
&\quad = \hat{U}^{-1}(\mathcal{M})\,\hat{D}[-(\mathbf{x}_0, \mathbf{k}_0)][\pi^{-n}\hat{D}[2(\mathbf{x}, \mathbf{k})]\exp(i\pi\hat{N})]\,\hat{D}[(\mathbf{x}_0, \mathbf{k}_0)]\,\hat{U}(\mathcal{M}) \\
&\quad = \pi^{-n}\hat{U}^{-1}(\mathcal{M})\,\hat{D}[-(\mathbf{x}_0, \mathbf{k}_0)]\,\hat{D}[2(\mathbf{x}, \mathbf{k})]\,\hat{D}[-(\mathbf{x}_0, \mathbf{k}_0)]\,\hat{U}(\mathcal{M})\exp(i\pi\hat{N}) \\
&\quad = \pi^{-n}\hat{U}^{-1}(\mathcal{M})\,\hat{D}[(2(\mathbf{x}-\mathbf{x}_0), 2(\mathbf{k}-\mathbf{k}_0))]\,\hat{U}(\mathcal{M})\exp(i\pi\hat{N}) \\
&\quad = \pi^{-n}\hat{U}^{-1}(\mathcal{M})\exp[i(\hat{\mathbf{q}}, \hat{\mathbf{p}})\,\mathcal{J}(2(\mathbf{x}-\mathbf{x}_0), 2(\mathbf{k}-\mathbf{k}_0))^{\mathsf{T}}]\,\hat{U}(\mathcal{M})\exp(i\pi\hat{N}) \\
&\quad = \pi^{-n}\exp[i(\hat{\mathbf{q}}, \hat{\mathbf{p}})(\mathcal{M}^{-1})^{\mathsf{T}}\,\mathcal{J}(2(\mathbf{x}-\mathbf{x}_0), 2(\mathbf{k}-\mathbf{k}_0))^{\mathsf{T}}]\exp(i\pi\hat{N}) \\
&\quad = \pi^{-n}\exp[i(\hat{\mathbf{q}}, \hat{\mathbf{p}})\,\mathcal{J}\mathcal{M}(2(\mathbf{x}-\mathbf{x}_0), 2(\mathbf{k}-\mathbf{k}_0))^{\mathsf{T}}]\exp(i\pi\hat{N}) \\
&\quad = \pi^{-n}\hat{D}[(2(\mathbf{x}-\mathbf{x}_0), 2(\mathbf{k}-\mathbf{k}_0))\,\mathcal{M}^{\mathsf{T}}]\exp(i\pi\hat{N}) \\
&\quad = \hat{\Delta}_W[(\mathbf{x}-\mathbf{x}_0, \mathbf{k}-\mathbf{k}_0)\,\mathcal{M}^{\mathsf{T}}]
\end{aligned} \tag{45}$$

we see that the unitary transformation of the quantum state made by an element in **WMp(2n, R)** induces an **ISp(2n, R)** transformation on the arguments $(\mathbf{x}, \mathbf{k})$ of the Wigner operator and Wigner function:

$$(\mathbf{x}', \mathbf{k}')^{\mathsf{T}} = \mathcal{M}(\mathbf{x}-\mathbf{x}_0, \mathbf{k}-\mathbf{k}_0)^{\mathsf{T}} \tag{46}$$

This induced inhomogeneous symplectic transformation is of the same form as (27), hence is also formally isomorphic to the quantum L.C.T. produced by the same unitary operator in (41), but note the exchange of orders between $\hat{T}\hat{U}$ and $\hat{U}^{-1}\hat{T}^{-1}$ from (41) to (45). The transformed Wigner function then takes the form

$$W_{\psi'}(\mathbf{x}, \mathbf{k}) = W_{\psi}(\mathbf{x}', \mathbf{k}') \tag{47}$$

Therefore, from the operator formulation of the Wigner function, we have obtained the corresponding **ISp(2n, R)** transformation on the Wigner function which corresponds to a given **WMp(2n, R)** transformation on the quantum state. It is clear that the relations derived above are independent of states as well as representations.

7. DISCUSSION

In this paper, we have shown that the quantum canonical transformation group $\mathbf{WMp(2n, R)} = \mathbf{W(2n)} \otimes_s \mathbf{Mp(2n, R)}$ is not isomorphic to but "larger than" its classical analogue $\mathbf{ISp(2n, R)} = \mathbf{T(2n)} \otimes_s \mathbf{Sp(2n, R)}$, since **W(2n)** has one more dimension than **T(2n)** which corresponds to a phase, and **Mp(2n, R)** is a doubly covering group of **Sp(2n, R)**. However, because both of the differences can be ascribed to a phase factor, and phase factors are always cancelled in the Wigner function, the transformation group on the Wigner function is exactly the classical **ISp(2n, R)** group.

This cancellation of phases in the Wigner function enables us to have a purely classical description of the Wigner function, i.e., we do not have to worry about the difference between **WMp(2n, R)** and **ISp(2n, R)** when we use the Wigner function to describe the semiclassical behavior of a quantum system.

In Ref. 4, the author discussed the use of the Wigner function as the representation of the one-mode squeezed states.[26,27] As a generalization, our results can be applied to the study of the Wigner functions of multimode squeezed coherent states[27,28] which correspond to $|\psi'\rangle$ in (43) with $|\psi\rangle = |0\rangle$. Furthermore, using (24), we can also apply the results in this paper to the Wigner functions of multimode thermal squeezed coherent states.[29]

From our operator formulation of the Wigner function, we can see that the Wigner operator is nothing but a special kind of elements of the n-dimensional **H(4)** group, i.e., the Lie group with its algebra spanned by $\{\hat{I}, \hat{q}_i, \hat{p}_j, \hat{N}\}$. Hence the Wigner function has a profound relation to this group. From the group-theoretic point of view, a possible generalization of the Wigner function is to consider the "generalized Wigner operators" corresponding to other groups.[30] This is a very interesting question and is worth pursuing further.

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